Problems and Solutions in Efficient, Accurate Computation of the Airy Functions in the Complex Plane

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Author’s Work

- Unpublished work (ARL/UT, 1969-1973) with Pedersen & Gordon (NUSC, 1972-1973), Honeywell Seattle (1976)
  – Produced normal mode models following Pedersen & Gordon’s landmark 1972 papers in JASA
  – Used Airy functions in ARL/UT normal mode models
  – Offered comments to NUSC that their modified Bessel functions of order 1/3 were proportional to Airy functions
  – Provided Padé approximant method for computing Airy functions in the complex plane
  – Dave Gordon produced his own coefficients and Padé sequence algorithms for his papers
- Recent Update
  – Noted that preliminary Digital Library of Mathematical Functions (DLMF) work on Airy functions did not have a section on the use of Padé approximants for Airy functions
  – Corresponded with author F.W.J. Olver who recommended that I publish my work
  – Revisited the problem with modern tools and resources

Three Topics for Today's Talk

- **Airy Functions**
  - **The most elementary of the special functions**
  - **Wave functions in boundary value problems**
- **Continued Fractions and Padé Approximants**
- **Practical Computation of Complex Functions near a Branch Cut of an Asymptotic Expansion**

Airy Functions in BIG

- **Airy functions of a complex argument and their first derivatives are used as wave functions in layered media, such as**
  - **Models of hydroacoustical propagation in the ocean**
  - **Predictions of eddy currents in plated waveguide inner walls**
  - **Transmission and reflection of light at oblique angles through coated lenses**
- **Major references**
  - **Airy Functions And Applications To Physics, Olivier Vallée and Manuel Soares, Imperial College Press (2004), ISBN-13: 978-1860944789.**
    - **http://www.maa.org/reviews/AiryFunction.html**
  - **Pedersen, M. A., and Gordon, D. F., Normal-Mode and Ray Theory Applied to Underwater Acoustic Conditions of Extreme Downward Refraction, *The Journal of the Acoustical Society of America*, v 51 No. 1, 1972 (two parts; see also Gordon's much larger paper in JASA in 1973).**
  - **Wikipedia http://en.wikipedia.org/wiki/Airy_Functions**

Definitions of Airy Functions

- **Solutions to Differential Equation**

$$f''(z) - z \cdot f(z) = 0$$

- **Normalization**

$$Ai(0) = \frac{1}{3^{2/3} \cdot \Gamma\left(\frac{2}{3}\right)}, \quad Ai'(0) = -\frac{1}{3^{1/3} \cdot \Gamma\left(\frac{1}{3}\right)}$$

$$Bi(0) = \frac{1}{3^{1/6} \cdot \Gamma\left(\frac{2}{3}\right)}, \quad Bi'(0) = \frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)}$$

Airy Functions as Bessel Functions

$$\zeta = \frac{2}{3} \cdot z^{\frac{3}{2}}$$

$$Ai(z) = \frac{1}{\pi} \cdot \sqrt{\frac{z}{3}} \cdot K_{\frac{1}{3}}(\zeta), \quad Ai'(z) = -\frac{1}{\pi} \cdot \frac{z}{\sqrt{3}} \cdot K_{\frac{2}{3}}(\zeta)$$

$$Bi(z) = \sqrt{\frac{z}{3}} \cdot \left(I_{\frac{1}{3}}(\zeta) + I_{-\frac{1}{3}}(\zeta) \right)$$

$$= \exp\left(j \cdot \frac{\pi}{6} \right) \cdot Ai\left(z \cdot \exp\left(j \cdot \frac{2\pi}{3} \right) \right)$$

$$+ \exp\left(-j \cdot \frac{\pi}{6} \right) \cdot Ai\left(z \cdot \exp\left(-j \cdot \frac{2\pi}{3} \right) \right)$$

Abramowitz & Stegun (10.4.6), (10.4.14) p. 377, 378

Small Argument Expansions

$$Ai(z) = c_1 \cdot f(z) - c_2 \cdot g(z)$$

$$Bi(z) = \sqrt{3} \cdot \left(c_1 \cdot f(z) + c_2 \cdot g(z)\right)$$

$$c_1 = Ai(0),\ \ c_2 = -Ai'(0)$$

$$f(z) = \sum_{k=0}^{\infty} 3^k \cdot \left(\frac{1}{3}\right)_k \cdot \frac{z^{3k}}{(3k)!}$$

$$g(z) = \sum_{k=0}^{\infty} 3^k \cdot \left(\frac{2}{3}\right)^k \cdot \frac{z^{3k+1}}{(3k+1)!}$$

Abramowitz & Stegun, (10.4.2), (10.4.3) p. 446

Large Argument Expansions

$$c_k = \frac{\Gamma\left(3k + \frac{1}{2}\right)}{54^k \cdot k! \cdot \Gamma\left(k + \frac{1}{2}\right)}, \;\; d_0 = 1, \;\; d_k = -\frac{6k+1}{6k-1} \cdot c_k$$

$$Ai(z) \sim \sqrt{\frac{4}{\pi}} \cdot z^{-\frac{1}{4}} \cdot \exp(-\zeta) \cdot \sum_{k=0} (-1)^k \cdot c_k \cdot \zeta^{-k}$$

$$Bi(z) \sim \sqrt{\frac{1}{\pi}} \cdot z^{-\frac{1}{4}} \cdot \exp(+\zeta) \cdot \sum_{k=0} c_k \cdot \zeta^{-k}$$

$$Ai'(z) \sim -\sqrt{\frac{1}{4\pi}} \cdot z^{\frac{1}{4}} \cdot \exp(-\zeta) \cdot \sum_{k=0} (-1)^k \cdot d_k \cdot \zeta^{-k}$$

$$Bi'(z) \sim \sqrt{\frac{1}{\pi}} \cdot z^{\frac{1}{4}} \cdot \exp(+\zeta) \cdot \sum_{k=0} d_k \cdot \zeta^{-k}$$

Existing Computation Methods

- **The Amos Method and Derivatives**
  - D. E. Amos, a portable package for Bessel functions of a complex argument and nonnegative order, ACM Transactions on Mathematical Software (TOMS), Volume 12 , Issue 3 (September 1986) pp 265-273, ISSN 0098-3500.
  - http://portal.acm.org/citation.cfm?id=214331
  - FORTRAN subroutines to compute Bessel and Airy functions in the complex plane
  - Accuracy losses up to half precision are tolerated before an error message is produced
- **Gaussian Quadrature – see Valée & Soares**
- **Taylor Series and Asymptotic Series**

Available in Math Packages

- **Mathematica**
  - http://mathworld.wolfram.com/AiryFunctions.html
- **MATLAB**
  - Uses the Amos method
  - Long link
- **Mathcad**
  - Algorithm not given
- **Others: limited to real argument**

Issues of Computational Methods

- Gaussian Quadrature – Efficiency
- Series
  - Word length – accuracy proportional to value of function divided by largest term in sum
  - Accuracy decrease is most severe when coefficients are alternating in arithmetic sign
- Recursion
  - Often the best approach
  - Amos methods based on use of robust recursion methods for Bessel functions, Airy functions in terms of Bessel functions of order 1/3
- Continued Fractions == Padé approximants
  - Are actually a form of recursion
  - Have numerical properties amenable to analysis
  - Provide the best solution in many problems
  - Are especially well-suited to functions of a complex argument

Why New Work is Important

- High accuracies with good speed are needed for modern applications such as
  - Numerical integration over frequency to model complex pulse shapes in multipath
  - Modeling signal processing performance for broadband signals requires accurate prediction of transfer function of the medium
  - Complex medium models using multiple layers use Airy functions in unpredictable regions of the complex plane
- Branch cuts in asymptotic expansions limit accuracy for moderate argument
- Numerical properties of small argument expansions limit accuracy in the same regions

Our Method: Continued Fractions

- Pioneered by Stieltjes at Toulouse and others in the 1890's
- Landmark work by H. S. Wall (1948)
- Commonly treated in mathematics until about 1948 – repeated division caused neglect in the early computer age?
- Shown early to be convergent over a wide range of conditions
- Most continued fractions from asymptotic series representing special functions converge to functions that have a branch cut ending at the origin

Common Continued Fraction Type

- **S-fraction or Stieltjes form (from Wall[1948])**

$$f_n(z) = \cfrac{1}{1 + \cfrac{c_1 \cdot z}{1 + \cfrac{c_2 \cdot z}{1 + \dots \cfrac{c_{n-1} \cdot z}{1 + c_n \cdot z}}}} = \frac{A_n}{B_n}$$

Recursion for the Convergents

- **Define a matrix of convergents**

$$MP_k = \begin{bmatrix} A_k & A_{k-1} \\ B_k & B_{k-1} \end{bmatrix}, \quad MP_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- **Recursion for this matrix**

$$MP_k = MP_{k-1} \cdot \begin{bmatrix} 1 & 1 \\ c_k \cdot z & 0 \end{bmatrix}, \; k > 0$$

- **Note equation for the determinant**

$$\left| MP_N \right| = \prod_{k=1}^{N} \left(-c_k \cdot z \right)$$

Classical Example: The Exponential Integral

$$E_n(x) = \int_1^\infty \frac{\exp(-z \cdot t)}{t^n} \cdot dt \sim \frac{\exp(-z)}{z} \cdot \sum_{k=0} \frac{(n)_k}{(-z)^k}$$

$$= \cfrac{\exp(-z)/z}{1+\cfrac{n/z}{1+\cfrac{1/z}{1+\cfrac{(n+1)/z}{1+\cfrac{2/z}{1+\cfrac{(n+2)/z}{1+\ldots}}}}}}$$

Abramowitz & Stegun pp 228-231

Another Classical Example

$$\ln\left(\Gamma\left(z\right)\right)=\tfrac{1}{2}\cdot\ln\left(2\pi\right)-z+\left(z-\tfrac{1}{2}\right)\cdot\ln\left(z\right)+\cfrac{a_0}{z+\cfrac{a_1}{z+\cfrac{a_2}{z+\cfrac{a_3}{z+\cfrac{a_4}{z+\cfrac{a_5}{z+\cfrac{a_6}{z+\cfrac{a_7}{z+\ldots}}}}}}}}$$

Coefficients in the Example

$$a_0 = \frac{1}{12},\ a_1 = \frac{1}{30},\ a_2 = \frac{53}{210},\ a_3 = \frac{195}{371},$$

$$a_4 = \frac{22999}{22737},\ a_5 = \frac{29944523}{19733142},\ a_6 = \frac{109535241009}{48264275462}$$

$$a_7 = \frac{29404527905795295658}{9769214287853155785}$$

From Abramowitz & Stegun 6.1.48 p. 258. Apparently Taken From H. S. Wall[1948], Eq. (93.9) on p. 365. The coefficient a_7 is first disclosed here

Canonic Form for Convergence Conditions

$$f_n = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \ldots \cfrac{1}{b_n}}}}}$$

Convergence Form for S-Fractions

$$f_n = \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{c_1 \cdot z} + \cfrac{1}{\cfrac{c_1}{c_2} + \cfrac{1}{\cfrac{c_2}{c_1 \cdot c_3 \cdot z} + \ldots}}}}$$

$$b_{2n+1} = \prod_{i=1}^{n} \frac{c_{2i-1}}{c_{2i}}, \;\; b_{2n} = \frac{1}{c_1 \cdot z} \cdot \prod_{i=1}^{n} \frac{c_{2i}}{c_{2i+1}}$$

Continued Fraction Convergence

- **Definition of convergence**

$$\lim_{n\to\infty} f_n = f$$

- **Stieltjes' Criteria**

$$\lim_{k\to\infty}\sum_{n=0}^{k} b_n \to \infty$$

- **Hamburger's Condition H: Any of**

$$\lim_{p\to\infty}\sum_{n=1}^{p}\left|b_{2n+1}\right| \to \infty$$

$$\lim_{p\to\infty}\left|\sum_{n=1}^{p} b_{2n}\right| \to \infty$$

$$\lim_{p\to\infty}\sum_{m=1}^{p}\left|b_{2m+1}\cdot\left(\sum_{n=1}^{m} b_{2n}\right)^{2}\right| \to \infty$$

Padé Approximants Near a Branch Cut

- **Both numerators and denominators represent sequences of orthogonal polynomials**
- **In any given approximant the poles and zeros alternate along the branch cut on the negative real axis**
- **Behavior of the approximants near a branch cut is similar to that of *tan(z)***

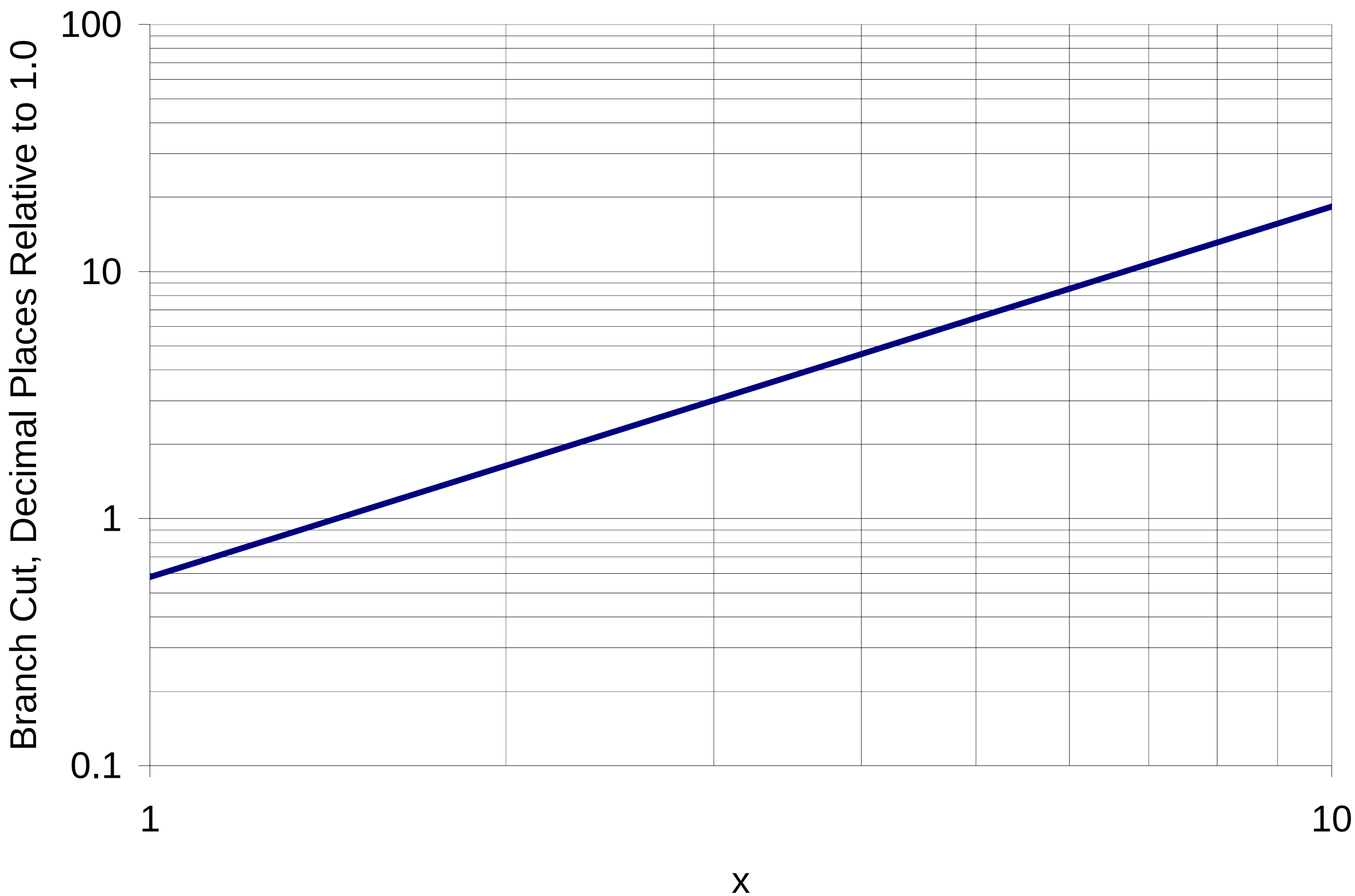
Behavior of Expansions for *Bi(z)*

- Small argument expansions are usually OK over wide regions in the complex plane
- Branch cut of large argument expansion is along the positive real axis
- Magnitude of step at point x is on the order of

$$\text{Step} \sim \sqrt{\frac{4}{\pi}} \cdot x^{-\frac{1}{4}} \cdot \exp\left(-\frac{2}{3} \cdot x^{\frac{3}{2}}\right)$$

- Accuracy is limited to about the magnitude of the step
- Solution:
  - Use a transformation to apply two expansions on real axis
  - Use small argument expansion to magnitudes that the branch cut "error" is acceptable as computation error

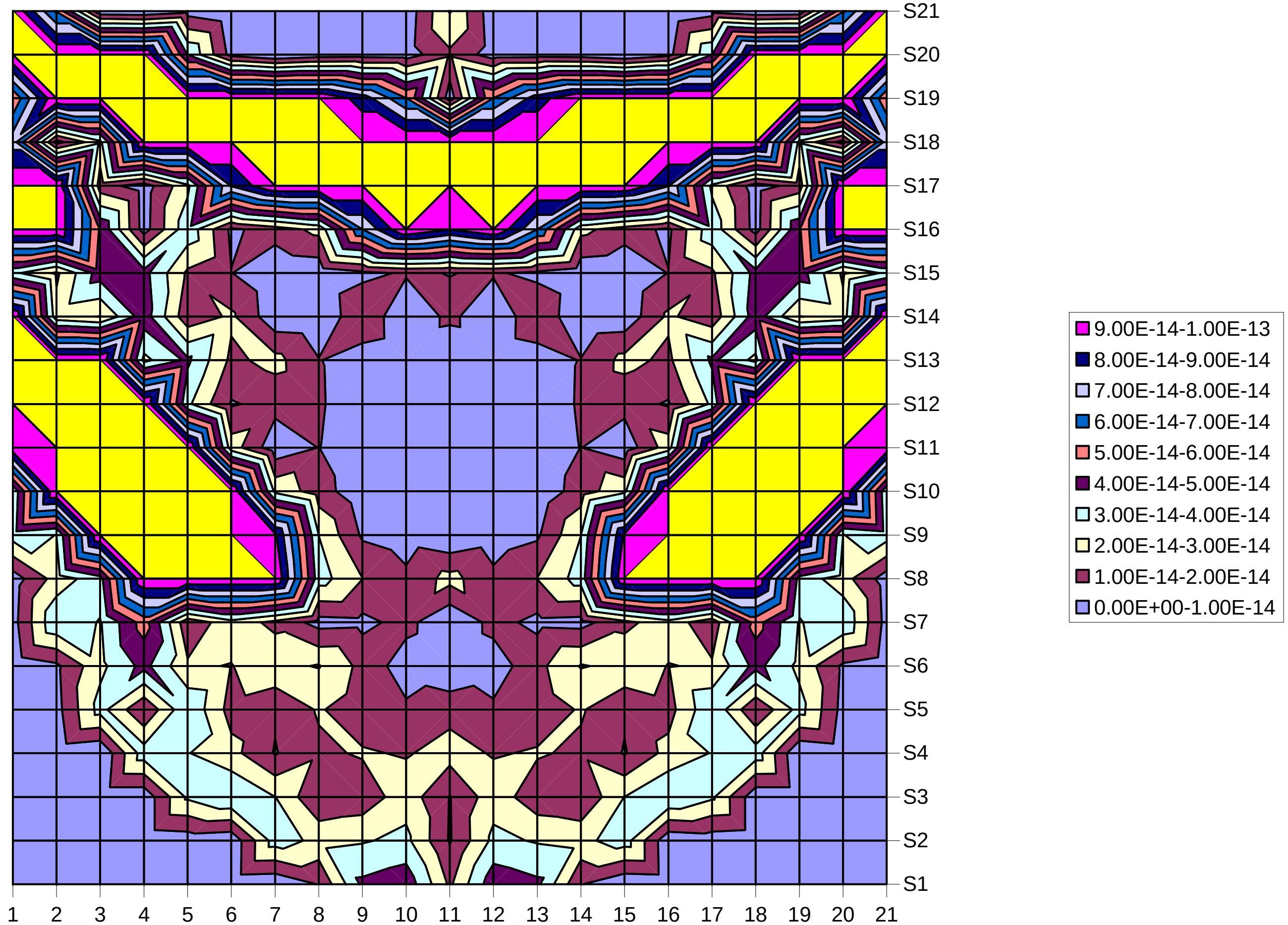
The Step Size of the Branch Cut Dies Out Exponentially



For *Bi(z)* Branch Cut is Parsed

- **Use odd and even coefficients of large argument expansion as separate continued fractions**
- **This gives us two functions that have branch cuts at angles of ±60°**
- **This allows computation of Bi(z) on the real axis for large argument**

Error Chart for *Bi(z)* for DP Continued Fractions



Solving the Remaining Problem

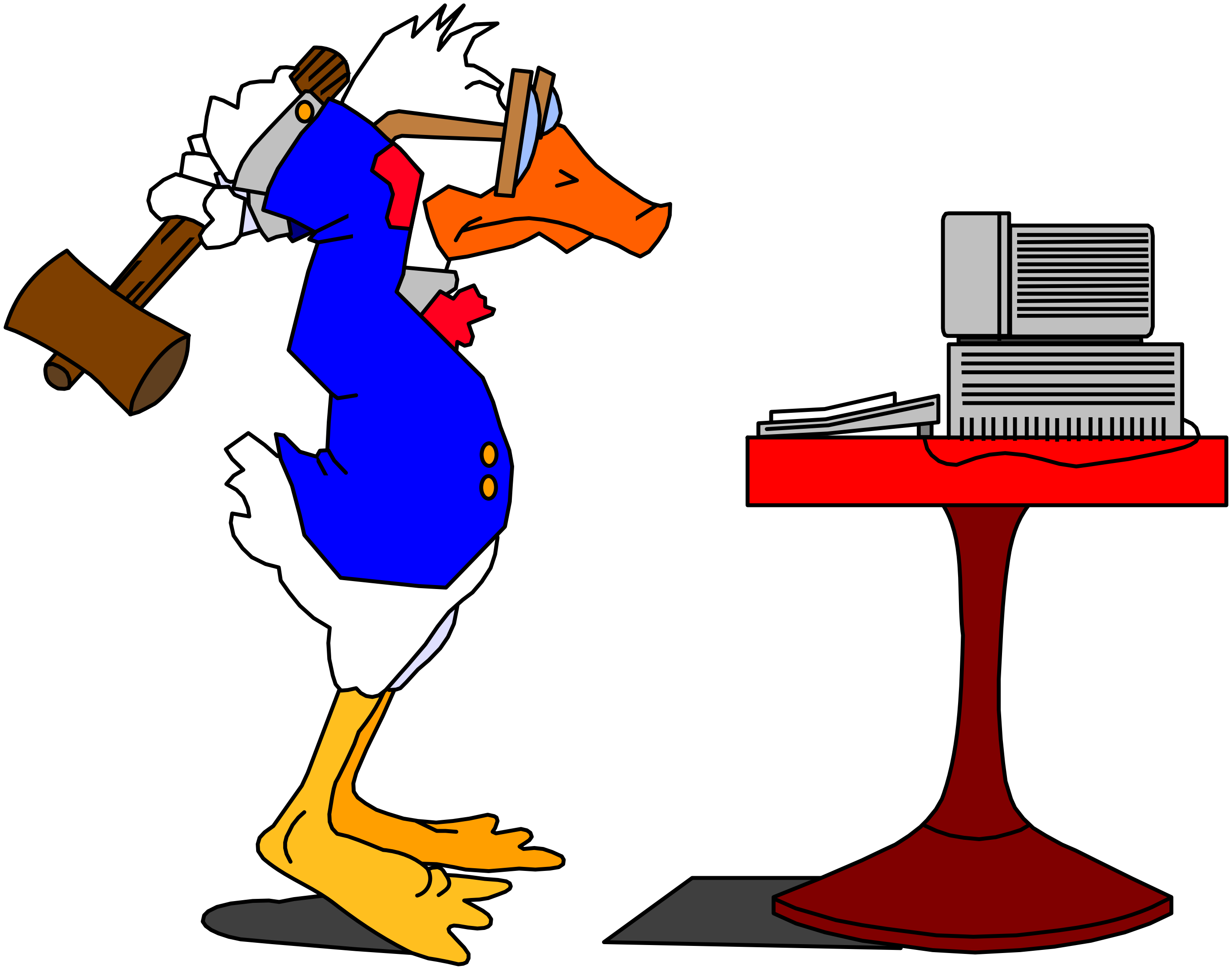
- **Add a Quad Precision Continued Fraction for the Trouble Regions**
- **Use fast quad precision in continued fractions**
  - **Quad precision is an intrinsic type in newer compilers**
  - **Arithmetic uses built-in carry and other support designed-in modern double precision hardware**
- **Use Even More Bits**
  - **For continued fraction coefficients**
  - **For normalization constants**

Conclusions

- **Need quad precision coefficients and constants**
- **These are available through multiple precision arithmetic**
- **Use fast quad precision in the difficult regions**
- **Use double precision hardware elsewhere**
- **Note that alternative large argument expansions are necessary for all the branch cuts**

Standard References

- Abramowitz & Stegun
  - Handbook of Mathematical Functions with Formulas Graphs and Mathematical Tables, Milton Abramowitz and Irene A. Stegun, National Bureau of Standards (1972), ISBN-10: 0318117304, ISBN-13: 978-0318117300
  - See also the Digital Library of Mathematical Functions, http://dlmf.nist.gov/
- Wall[1948]
  - Analytic Theory of Continued Fractions, American Mathematical Society (January 2000), reprint of Chelsea 1948 edition, ISBN-10: 0821821067, ISBN-13: 978-0821821060
- DLMF, http://dlmf.nist.gov/ (Airy functions not up as of 12/30/07)
- D. E. Amos' methods, http://www.netlib.org/amos/
- See also the thread of literature and books on Continued Fractions



Continued Fraction Coefficients and Constants

```
!Ai(0)
  real(kind(0d0)),private::Ai_0= 3.5502805388781760D-01
!Aiprime(0)
  real(kind(0d0)),private::Ai_prime_0=-2.5881940379280730D-01
!Bi(0)
  real(kind(0d0)),private::Bi_0= 6.1492662744600140D-01
!Biprime(0)
  real(kind(0d0)),private::Bi_prime_0= 4.4828835735382710D-01
!Small argument Ai(z) f(z)
  integer,parameter,private::n_Aif=50
  real(kind(0d0)),dimension(50),private::c_Aif=(/           &
    -1.6666666666666670D-01, 1.3333333333333330D-01,-4.8611111111111120D-03,        &
     1.9796176046176050D-02,-1.6961019238923560D-03, 7.4811855133535110D-03,        &
    -8.6267827469819790D-04, 3.8951201773771840D-03,-5.2135577927749650D-04,        &
     2.3825532162205740D-03,-3.4886020918362140D-04, 1.6061039847626790D-03,        &
    -2.4972153536511930D-04, 1.1554670154203870D-03,-1.8754501855265490D-04,        &
     8.7093156736240310D-04,-1.4600097140192230D-04, 6.7987480543480220D-04,        &
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     4.4723930780172290D-04,-7.9752956703822110D-05, 3.7336331214849910D-04,        &
    -6.7501524090580320D-05, 3.1638651119734550D-04,-5.7870767540383880D-05,        &
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    -3.0815195232937060D-05, 1.4507537055804570D-04,-2.7735573190063230D-05,        &
     1.3063823996833570D-04,-2.5095562615290610D-05, 1.1825360546792950D-04,        &
    -2.2815304745193720D-05, 1.0755007369551070D-04,-2.0832284269241070D-05,        &
     9.8236605350344650D-05,-1.9096998135884000D-05, 9.0082521273373250D-05,        &
    -1.7569835694598920D-05, 8.2903080716075160D-05/)
```

```
!Small argument Ai(z) g(z)
  integer,parameter,private::n_Aig=50
  real(kind(0d0)),dimension(50),private::c_Aig=(/           &
    -8.3333333333333320D-02, 5.9523809523809520D-02,-5.0793650793650790D-03,         &
     1.3664529914529910D-02,-1.7322477496441530D-03, 5.8945369927128760D-03,         &
    -8.7221300261986110D-04, 3.2676982093060010D-03,-5.2486370928664210D-04,         &
     2.0732998253881880D-03,-3.5043615313818260D-04, 1.4315594983302830D-03,         &
    -2.5053057111190160D-04, 1.0474787725873700D-03,-1.8800188030287290D-04,         &
     7.9953579034192600D-04,-1.4627807415140270D-04, 6.3024040617195520D-04,         &
    -1.1705312713972020D-04, 5.0953226721521540D-04,-9.5789087724439930D-05,         &
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    -3.0827572162112640D-05, 1.4001935771452760D-04,-2.7745600836549920D-05,         &
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    -2.2822091272061380D-05, 1.0430994550560390D-04,-2.0837942735590360D-05,         &
     9.5405009916099490D-05,-1.9101753483443010D-05, 8.7593565883266250D-05,         &
    -1.7573861123049990D-05, 8.0703639549915760D-05/)
!Small argument Aiprime(z) F(z)
  integer,parameter,private::n_Aifd=49
  real(kind(0d0)),dimension(49),private::c_Aifd=(/          &
    -6.6666666666666670D-02, 4.5833333333333330D-02,-4.8783287419651050D-03,         &
     1.1706059834402080D-02,-1.7142783283385870D-03, 5.3026331015274830D-03,         &
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     9.4040430026170850D-05,-1.9100111804183420D-05, 8.6392256041740120D-05,         &
    -1.7572473900933200D-05/)
```

```
!Small argument Aiprime(z) G(z)
  integer,parameter,private::n_Aigd=49
  real(kind(0d0)),dimension(49),private::c_Aigd=(/          &
    -3.333333333333330D-01, 2.916666666666670D-01,-3.684807256235828OD-03,        &
     2.483865341008198OD-02,-1.613367905502737OD-03, 8.569497713195044OD-03,        &
    -8.443352028025023OD-04, 4.289946896079833OD-03,-5.151949877879773OD-04,        &
     2.567804879767453OD-03,-3.462398850971784OD-04, 1.707379031591397OD-03,        &
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     1.328963459420662OD-04,-2.508400345481785OD-05, 1.201968743681144OD-04,        &
    -2.280577559357023OD-05, 1.092344162061942OD-04,-2.082435840433440OD-05,        &
     9.970605211515775OD-05,-1.909035209490513OD-05, 9.137212777310445OD-05,        &
    -1.756422126222039OD-05/)
!1/sqrt(pi)
  real(kind(0d0)),private::c00= 5.641895835477563OD-01
!1/(2*sqrt(pi))
  real(kind(0d0)),private::c002= 2.820947917738781OD-01
```

```
!Large positive argument Ai(z)
  integer,parameter,private::n_Aipl=50
  real(kind(0d0)),dimension(50),private::c_Aipl=(/         &
     6.944444444444450D-02, 4.652777777777780D-01, 5.613253178551686D-01,        &
     9.623713646092863D-01, 1.057903488622767D+00, 1.461186750970490D+00,        &
     1.555888096882571D+00, 1.960527343591469D+00, 2.054514724429398D+00,        &
     2.460102722507219D+00, 2.553498495288027D+00, 2.959804798938829D+00,        &
     3.052705545297543D+00, 3.459583534767610D+00, 3.552063437479150D+00,        &
     3.959412394304397D+00, 4.051529035294769D+00, 4.459275923343153D+00,        &
     4.551074801471019D+00, 4.959164479985826D+00, 5.050682202512990D+00,        &
     5.459071721995747D+00, 5.550338234767595D+00, 5.958993298818395D+00,        &
     6.050033462447937D+00, 6.458926122468009D+00, 6.549760846005236D+00,        &
     6.958867938094291D+00, 7.049515009768522D+00, 7.458817059331949D+00,        &
     7.549291766367966D+00, 7.958772198887988D+00, 8.049087797758515D+00,        &
     8.458732356530795D+00, 8.548900435251838D+00, 8.958696742938708D+00,        &
     9.048727504120457D+00, 9.458664726655389D+00, 9.548567211471106D+00,        &
     9.958635796343195D+00, 1.004841806381116D+01, 1.045860953340928D+01,        &
     1.054827880542635D+01, 1.095858559181569D+01, 1.104814837162218D+01,        &
     1.145856368296021D+01, 1.154802585276269D+01, 1.195854356419813D+01,        &
     1.204791046627328D+01, 1.245852503001880D+01/)
!Large positive argument Ai_prime(z)
  integer,parameter,private::n_Aidpl=49
  real(kind(0d0)),dimension(49),private::c_Aidpl=(/        &
    -9.722222222222223D-02, 5.486111111111112D-01, 4.247245663384904D-01,        &
     1.043885299132362D+00, 9.324186359175586D-01, 1.542641569990111D+00,        &
     1.436562719654159D+00, 2.042110806678425D+00, 1.939228729435063D+00,        &
     2.541833014436892D+00, 2.441121130150114D+00, 3.041669892085209D+00,        &
     2.942551176719093D+00, 3.541566793733079D+00, 3.443679704419501D+00,        &
     4.041498273648775D+00, 3.944599048918660D+00, 4.541451087723099D+00,        &
     4.445366404800471D+00, 5.041417761013081D+00, 4.946019297782122D+00,        &
     5.541393805835297D+00, 5.446583485446427D+00, 6.041376393378154D+00,        &
     5.947077302693627D+00, 6.541363668748419D+00, 6.447514197563693D+00,        &
     7.041354375392468D+00, 6.947904284399491D+00, 7.541347638502184D+00,        &
     7.448255334281063D+00, 8.041342834725434D+00, 7.948573428545389D+00,        &
     8.541339510933851D+00, 8.448863402813824D+00, 9.041337331991750D+00,        &
     8.949129156445332D+00, 9.541336046260318D+00, 9.449373873055626D+00,        &
     1.004133546227290D+01, 9.949600180781294D+00, 1.054133543263224D+01,        &
     1.044981027080903D+01, 1.104133584268552D+01, 1.095000598642302D+01,        &
     1.154133660242637D+01, 1.145018889085488D+01, 1.204133764061680D+01,        &
     1.195036031964765D+01/)
```

Coefficients increase linearly with index

```
!Large negative real part, even coefficients
  integer,parameter,private::n_Ainlc=50
  real(kind(0d0)),dimension(50),private::c_Ainlc=(/         &
     3.7133487654320980D-02, 1.5153517232510290D+00, 3.5914488357300260D+00,        &
     7.8539243889472670D+00, 1.2090856886865960D+01, 1.9107040598853890D+01,        &
     2.5531064540092010D+01, 3.5284947833322160D+01, 4.3910246534350480D+01,        &
     5.6391336074081630D+01, 6.7227389832514440D+01, 8.2428028945395830D+01,        &
     9.5481856114167180D+01, 1.1339608923165770D+02, 1.2867320930493640D+02,        &
     1.4929620296289340D+02, 1.6680113417244510D+02, 1.9012884497288210D+02,        &
     2.0986539310217100D+02, 2.3589436100932710D+02, 2.5786580111533220D+02,        &
     2.8659301271223700D+02, 3.1080221047056010D+02, 3.4222500415481690D+02,        &
     3.6867450067997780D+02, 4.0279049843804670D+02, 4.3148257176547770D+02,        &
     4.6828962855893050D+02, 4.9922633954706780D+02, 5.3872250480425570D+02,        &
     5.7190573225617110D+02, 6.1408921994288210D+02, 6.4952068804166590D+02,        &
     6.9438985297151900D+02, 7.3207115309460130D+02, 7.7962447188015170D+02,        &
     8.1955708021221610D+02, 8.6979313573507540D+02, 9.1197842768061630D+02,        &
     9.6489589627575260D+02, 1.0093351583929710D+03, 1.0649327991581300D+03,        &
     1.1116272391447180D+03, 1.1699038849361670D+03, 1.2188546400637980D+03,        &
     1.2798091898463500D+03, 1.3310173341452960D+03, 1.3946487464417730D+03,        &
     1.4481152968677940D+03, 1.5144225841098340D+03/)
!Large negative real part, odd coefficients normalization
  real(kind(0d0)),private::c0_Ainls= 6.9444444444444450D-02
!Large negative real part, odd coefficients
  integer,parameter,private::n_Ainls=50
  real(kind(0d0)),dimension(50),private::c_Ainls=(/         &
     5.4710005144032920D-01, 2.5086966306584360D+00, 5.4860460679287040D+00,        &
     9.9315792367643250D+00, 1.5355723514035650D+01, 2.2282583627216730D+01,        &
     3.0159397203884890D+01, 3.9565007215628140D+01, 4.9897711956702990D+01,        &
     6.1780140733087160D+01, 7.4570861377719260D+01, 8.8928632260020740D+01,        &
     1.0417891290118830D+02, 1.2101085749428240D+02, 1.3872188960726280D+02,        &
     1.5802705568992180D+02, 1.7819979693431700D+02, 1.9997738966660010D+02,        &
     2.2261263288348930D+02, 2.4686197584201190D+02, 2.7196039233225610D+02,        &
     2.9868090070106830D+02, 3.2624306897893110D+02, 3.5543423048021500D+02,        &
     3.8546065624993130D+02, 4.1712201718483830D+02, 4.4961314772443720D+02,        &
     4.8374430249910780D+02, 5.1870053732490160D+02, 5.5530112042213930D+02,        &
     5.9272281939109910D+02, 6.3179249910159640D+02, 6.7167998869588200D+02,        &
     7.1321846214340420D+02, 7.5557204043228210D+02, 7.9957902956897240D+02,        &
     8.4439897018737020D+02, 8.9087421852894530D+02, 9.3816077391094340D+02,        &
     9.8710404384486770D+02, 1.0368574478832460D+03, 1.0882685184267320D+03,        &
     1.1404889886838390D+03, 1.1943676535993740D+03, 1.2490553931626150D+03,        &
     1.3054014593608380D+03, 1.3625566584133460D+03, 1.4213699445892150D+03,        &
     1.4809927817498130D+03, 1.5422731172099560D+03/)
```

```
!Derivative, large negative real part, even coefficients
  integer,parameter,private::n_Aindlc=50
  real(kind(0d0)),dimension(50),private::c_Aindlc=(/        &
    -4.388503086419753OD-02, 1.4717560442386830D+00, 3.3873177172351710D+00,         &
     7.8475018652350190D+00, 1.1802258356126150D+01, 1.9143722777557620D+01,         &
     2.5169852029282360D+01, 3.5366683307851920D+01, 4.3482521271895000D+01,         &
     5.6518886369900960D+01, 6.6736983956286650D+01, 8.2601681935119570D+01,         &
     9.4931452463884690D+01, 1.1361591251075580D+02, 1.2806481929682170D+02,         &
     1.4956215227953940D+02, 1.6613633861644850D+02, 1.9044081652243880D+02,         &
     2.0914547780891570D+02, 2.3625221881050050D+02, 2.5709183960879260D+02,         &
     2.8699660379475650D+02, 3.0997511757937240D+02, 3.4267416733852720D+02,         &
     3.6779506894336920D+02, 4.0328506955048730D+02, 4.3055149712549470D+02,         &
     4.6882944358624860D+02, 4.9824424005752830D+02, 5.3930740180152160D+02,         &
     5.7087316206883840D+02, 6.1471904017974760D+02, 6.4843814809764270D+02,         &
     6.9506444159701550D+02, 7.3093909945536870D+02, 7.8034367828075730D+02,         &
     8.1837593066053490D+02, 8.7055681369543850D+02, 9.1074856702760640D+02,         &
     9.6570390401215070D+02, 1.0080569428004680D+03, 1.0657849992702770D+03,         &
     1.1103009996863560D+03, 1.1708001443074260D+03, 1.2174806856892960D+03,         &
     1.2807493795123480D+03, 1.3295959541709830D+03, 1.3956327414407930D+03,         &
     1.4466467630867660D+03, 1.5154502633239510D+03/)
!Derivative, large negative real part, odd coefficients normalization
  real(kind(0d0)),private::c0_Aindls=-9.7222222222222230D-02
!Derivative, large negative real part, odd coefficients
  integer,parameter,private::n_Aindls=50
  real(kind(0d0)),dimension(50),private::c_Aindls=(/        &
     4.3676054526748970D-01, 2.4859342849794240D+00, 5.2837945434141480D+00,         &
     9.9322755147524160D+00, 1.5085342473820170D+01, 2.2313603376479640D+01,         &
     2.9830023128126380D+01, 3.9628836983304710D+01, 4.9514362941462630D+01,         &
     6.1877915884484040D+01, 7.4136781261513890D+01, 8.9060922060383410D+01,         &
     1.0369640620965820D+02, 1.2117795243252990D+02, 1.3819269772189780D+02,         &
     1.5822909455962950D+02, 1.7762529410342310D+02, 2.0021442307987800D+02,         &
     2.2199393909450070D+02, 2.4713400081229020D+02, 2.7129844324808660D+02,         &
     2.9898788070031790D+02, 3.2553866177127720D+02, 3.5577610764427640D+02,         &
     3.8471448099987390D+02, 4.1749872002287810D+02, 4.4882580972421690D+02,         &
     4.8415575090755510D+02, 5.1787257339694830D+02, 5.5574722902075350D+02,         &
     5.9185471013284790D+02, 6.3227317949225240D+02, 6.7077216786650030D+02,         &
     7.1373362445910110D+02, 7.5462490228275230D+02, 8.0012858354495360D+02,         &
     8.4341287527761320D+02, 8.9145807424614670D+02, 9.3713605379223280D+02,         &
     9.8772211224524020D+02, 1.0357944089149540D+03, 1.0889207116677260D+03,         &
     1.1393879151796760D+03, 1.1950538852938480D+03, 1.2479165500104220D+03,         &
     1.3061216447346930D+03, 1.3613802932764650D+03, 1.4221240005796020D+03,         &
     1.4797791269321980D+03, 1.5430609625203890D+03/)
```