Interactive Multiple Models

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Introduction

- Interactive multiple models (IMM)
  - Used in Kalman filter
    - » Extrapolation from last update time to current radar measurement time using target motion model
    - » Update using estimation theory
  - Multiple models used in extrapolation
- IMMs improve accuracy of Kalman filter
- Theory of IMMs is based on
  - Discrete Markov processes
  - Additional estimation theory

Today's Topics

- Functions in an IMM
  - Discrete Markov models
  - Bayesian update of the probability vector
  - Definition of a single state vector and covariance matrix from multiple models and the probability vector
- Equal time for the three topics
- Examples – aircraft, MECO

Overview

- Tracker update consists of
  - Extrapolation of target position, velocity, and tracker errors from last update time to current radar measurement time
  - Correlation or association – match track files to radar returns
  - Update the track file with the radar data
- IMM allow
  - Use of more than one target motion model
  - Improved performance

Discrete Markov Process

- A discrete Markov process is based on the concept of a probability vector
- A probability vector is a set of probabilities that a system is in each of a set of mutually exclusive states
- A probability vector $\underline{p}_i$ can be propagated to another probability vector $\underline{p}_{i+1}$ by a linear transformation:

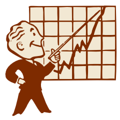
$$\underline{p}_{i+1} = M \cdot \underline{p}_i$$

Examples of a Discrete Markov Process

- Terrain obscuration
  - Terrain is modeled as random
  - Specify the probability that a clear line of sight will become obscured in a given time
  - Specify the probability that an obscured line of sight will become clear in a given time
- MECO
- Aircraft motion
  - Random maneuvering
  - Hard turn

Variation of a Probability Vector With Time

- Define the probability that the system will change from state "j" to state "i" in time Δt as Δa_{ij}
- The probability that the system will remain in state "j" in time Δt is



$$\Delta a_{jj} = 1 - \sum_{j \neq i} \Delta a_{ij}$$

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The Linear Transformation

- Define a matrix M

$$M(\Delta t) = [\Delta a_{ij}]$$

- The probability vector obeys the linear transformation

$$\underline{p}(t + \Delta t) = M \cdot \underline{p}(t)$$

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Properties of The Markov Matrix

- Columns are probability vectors
- No characteristic value can exceed 1.0
- When all elements of M are positive
  - One and only one characteristic value of M exists that is equal to +1.0
  - The corresponding characteristic vector is a positive probability vector

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The Summation Operator

- A vector of all ones is a summation operator

$$\underline{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

- Dot or inner product of summation operator with probability vector is one

$$\underline{1}^T \cdot \underline{p} = 1$$

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Markov Matrices

- Left-multiply by transpose of summation operator

$$\underline{1}^T \cdot M = \underline{1}^T$$

- Converse: if all elements of M are nonnegative and this equation holds, M is a Markov matrix
- It follows that the product of two Markov matrices is a Markov matrix

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Examples of Markov Matrices

- The terrain obscuration example

$$M = \begin{bmatrix} 1 - P(Obsc|Clear) & P(Clear|Obsc) \\ P(Obsc|Clear) & 1 - P(Clear|Obsc) \end{bmatrix}$$

- The two-state toggle

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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Characteristic Values

- For terrain obscuration example

$$\lambda = 1,\ 1 - P(Clear|Obsc) - P(Obsc|Clear)$$

- For two-state toggle example

$$\lambda = +1,\ -1$$

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Characteristic Vectors

- Terrain obscuration example

$$\underline{p}_1 = c \cdot \begin{bmatrix} P(Clear|Obsc) \\ P(Obsc|Clear) \end{bmatrix},\ \underline{1}^t \cdot \underline{p}_{1-P_C-P_O} = 0$$

  - Second characteristic value <1 when M is positive
  - Second characteristic vector not a probability vector
- Two-state toggle example
  - Stationary points, not limiting vectors except in an averaging sense

$$\underline{p}_{+1} = \begin{bmatrix} .5 \\ .5 \end{bmatrix},\ \underline{p}_{-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{(not a probability vector)}$$

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Limiting Values

- Terrain obscuration example

$$\underline{p} = \frac{1}{P(Obsc|Clear) + P(Clear|Obsc)} \cdot \begin{bmatrix} P(Clear|Obsc) \\ P(Obsc|Clear) \end{bmatrix}$$

- Two-state toggle example
  - No limiting value independent of initial conditions
  - "Mean" limiting value does exist

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Continuous Stochastic Process

- Define the matrix A as

$$A = \lim_{\Delta t \to 0} \frac{M(\Delta t)}{\Delta t}$$

- When $\underline{p}_0$ is a probability vector a continuous probability vector is given by

$$\dot{\underline{p}} = A \cdot \underline{p},\ \ \underline{p}(t_0) = \underline{p}_0$$

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Solution for Time Invariant Case

- Probability vector versus time

$$\underline{p}(t) = \exp(A \cdot (t - t_0)) \cdot \underline{p}(t_0)$$

- Definition of matrix exponential

$$M(dt) = \exp(A \cdot dt) = I + \sum_{i=1}^{\infty} \frac{1}{i!} \cdot [A \cdot dt]^i$$

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Bayesian Update of the Probability Vector

- Kalman filter with IMM
  - State vector and error covariance extrapolates from last update time to current time
  - Probability vector for target in each of K states extrapolated from last update time to current time using a Markov matrix
- Remaining operations to complete the update
  - Update the probability vector
  - Recombine the state vectors

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Use Association Information to Update the Probability Vector

- Method – weight each probability with its association likelihood and renormalize
- Begin with the extrapolated probability vector $\underline{\tilde{p}}(t) = \exp\left(A \cdot \left(t - t_{\_}\right)\right) \cdot \underline{p}\left(t_{\_}\right)$
- Use the likelihood la_j from each of the associations

$$la_j = \frac{1}{(2\pi)^{K/2} \cdot \left|E_j\right|^{1/2}} \cdot \exp\left(-\frac{1}{2} \cdot \underline{e}_j^T \cdot E_j^{-1} \cdot \underline{e}_j\right), \quad j = 1 \ldots K$$

$$\underline{e}_j = \underline{y} - \underline{h}\left(\underline{\tilde{x}}_j\right), \quad E_j = H \cdot \tilde{P}_j \cdot H^T + R$$

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Methodology for Update of Probability Vector

- Updated probability vector using Bayes' theorem

$$p_k(t) = P\left(k \middle| \underline{y}\right) = \frac{P\left(\underline{y} \middle| k\right) \cdot \tilde{p}_k(t)}{\sum_k P\left(\underline{y} \middle| k\right) \cdot \tilde{p}_k(t)} = \frac{la_k \cdot \tilde{p}_k(t)}{\sum_k la_k \cdot \tilde{p}_k(t)}$$

- The updated probability vector is

$$\underline{p}_U(t) = \begin{bmatrix} \vdots \\ la_j \cdot \tilde{p}_j \\ \vdots \end{bmatrix}, \quad \underline{p}(t) = \frac{1}{\underline{1}^T \cdot \underline{p}_U(t)} \cdot \underline{p}_U(t)$$

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Unifying the State Vector

- We have K state vectors, covariance matrices, and probabilities
- The system is in only one of K states
- Unification: use the Bayesian mean

$$\underline{\hat{x}}(t) = \sum_k P\left(k \middle| \underline{y}\right) \cdot \underline{\hat{x}}_k = \sum_k p_k(t) \cdot \underline{\hat{x}}_k$$

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Unifying the Covariance Matrix

- Covariance matrix follows from Bayesian mean for state vector

$$P(t) = \left\langle \left(\underline{\hat{x}} - \underline{x}\right) \cdot \left(\underline{\hat{x}} - \underline{x}\right)^T \right\rangle$$
$$= \sum_k p_k(t) \cdot P_k + \sum_k p_k(t) \cdot \left(\underline{\hat{x}}_k - \underline{\hat{x}}\right) \cdot \left(\underline{\hat{x}}_k - \underline{\hat{x}}\right)^T$$

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MECO

- MECO is main engine cutoff
- This is an obvious candidate – three target motion models
  - No MECO
  - MECO
  - Existing model encompassing both cases

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MECO System Models

- MECO has not occurred
  - Process noise is low
  - System model has acceleration along velocity vector
- MECO has occurred about time of last update
  - Process noise is very low
  - Gravity acceleration only
- MECO occurred sometime since last update
  - This is the current model, unmodified
  - Process noise high
  - Intermediate acceleration along velocity vector

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IMM Benefits for the MECO Problem

- Lower process noise
  - Every update but one uses either MECO or non-MECO model with lower process noise
- Enhanced performance
  - Lower process noise allows better association performance
  - Lower process noise provides lower tracker errors
  - Better tracker accuracy provides better association gate accuracy

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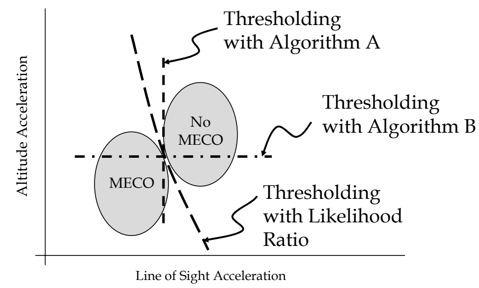
Determining MECO



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MECO Determination

- Likelihood ratio uses
  - All the measurements
  - All the states
  - The covariance matrix
- Simplest – and best performance
  - Implement in the measurement space
  - Minimize computation

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MECO Determination Using the Probability Vector

- The probability vector is an indicator of when MECO occurs
- The probability vector combines propagation using best estimate of likelihood of MECO as a function of time – the A matrix
- The Bayesian update of the probability vector implements a likelihood ratio test in the measurement space
- Conclusion: IMM can provide excellent performance in MECO determination

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Aircraft Motion

- Example – Singer's aircraft motion model
  - No maneuver, probability P_1
  - Hard turn left, acceleration A, probability $P_2/2$
  - Hard turn right, acceleration A, probability $P_2/2$
  - Random lateral acceleration, probability P_3

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Process Noise for Each Case

- Non-maneuvering: zero
- Hard turn left: zero
- Hard turn right: zero
- Random maneuvering: $A^2/6$
- Compares to single model: $A^2 \cdot (P_2+P_3/6)$
- Result: IMM provides improved performance

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Summary

- Keys to success with IMM
  - Selection of system models for distinct, observable differences at association time
  - The Markov matrix
- Use *a priori* information
  - Target type
  - CONOPS and mission timeline
- Keep number of models under 4
  - Avoid information overload

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