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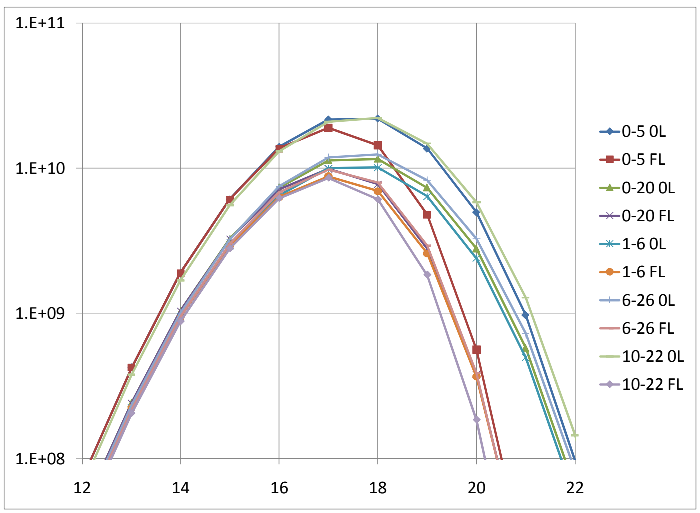
Costas array search technique that maximizes backtrack and symmetry exploitation



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Abstract

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- Costas search techniques
  - Generators don't find all of them
  - We present two innovations that improve speed
- Innovations presented here
  - Essentially full exploitation of symmetry
  - Multiple level look-ahead preclusion
- Advantages gained
  - Well-known symmetry gains a factor of two
  - New symmetry gains approach a factor of four
  - Look-ahead gains approach a factor of two
  - Overall, factor of four over older methods

Topics Today

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- Background
  - Backtrack programming with preclusion
    - The difference table
    - Generating the preclusion table
  - Symmetry and sets related by rotation and transposition
    - Sets of eight Costas arrays, when asymmetrical
    - Symmetry causes duplications; only four to a set
- Innovation: full symmetry exploitation
- Innovation: multi-level look-ahead preclusion
- Results
  - Overall a factor of four improvement
  - The Last Costas Array?
- Costas arrays of large orders

The Difference Table

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- Example: {4,2,5,1,3}
  - Row *nr* is difference between columns *nc+nr* and *nc*, given as *D<nr>(nc)=CA(nc+nr)-CA(nc)*

Col	**1**	**2**	**3**	**4**	**5**
CA	4	2	5	1	3
D1	-2	3	-4	2	
D2	1	-1	-2		
D3	-3	1			
D4	-1				

The Preclusion Table

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- Getting Started: Given two column indices

Col	1	2	3	4	5
CA	4	2	5	1	3
D1	-2	3	-4	2	
D2	1	-1	-2		
D3	-3	1			
D4	-1				

- Use the one available difference to preclude values of the third column index that would result in a duplication

How the Preclusion Table Works

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- Begin with the difference matrix

Col	1	2	3	4	5
CA	4	2	5	1	3
D1	-2	3	-4	2	
D2	1	-1	-2		
D3	-3	1			
D4	-1				

- Note that D1(2) is

$$D1(2) = CA(3) - CA(2) \neq D1(1)$$

- So, we have

$$CA(3) \neq D1(1) + CA(2)$$

The First Preclusion Table

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- Table of precluded values of third column index

Col	1	2	3	4	5
CA	4	2	5	1	3
D1	-2	3	-4	2	
D2	1	-1	-2		
D3	-3	1			
D4	-1				

Row Index	Reason
0	D1(1)+CA(2)
1	
2	Taken
3	
4	Taken
5	

The Second Preclusion Table

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- Table of precluded values of fourth column index

Col	1	2	3	4	5
CA	4	2	5	1	3
D1	-2	3	-4	2	
D2	1	-1	-2		
D3	-3	1			
D4	-1				

Row Index	Reason
2	Taken
3	D1(1)+CA(3), D2(1)+CA(2)
4	Taken
5	Taken
6	
7	
8	D1(2)+CA(3)

The Third Preclusion Table

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- Table of precluded values of the fifth column index

Col	1	2	3	4	5
CA	4	2	5	1	3
D1	-2	3	-4	2	
D2	1	-1	-2		
D3	-3	1			
D4	-1				

Row Index	Reason
-3	D1(3)+CA(4)
-2	
-1	D1(1)+CA(4), D3(1)+CA(2)
0	
1	Taken
2	Taken
3	
4	Taken, D1(2)+CA(4), D2(2)+CA(3)
5	Taken
6	D2(1)+CA(3)

Bitmasks for Rows of the Difference Matrix

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- The difference table consists of numbers that may be anywhere from –(N-1) to (N-1)
- The Costas condition is that no entry is ever allowed to repeat
  - Bit 32 of a 64-bit register represents a difference of zero
  - Bit positions in a 64-bit register can be used to represent a row of a difference matrix for searches up to order 33

Bitmask for the Preclusion Table

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- The preclusion table is shown for all values
- Only row indices from 0 to N-1 are significant in the implementation
  - A 32-bit mask is sufficient for searches up to order 32
  - Initialize with the rows used up to that point in the search
- For each available row of the difference matrix
  - Shift the difference mask by the row indices
  - Update the preclusion mask with a logical OR

Architecture-Independent Efficiency Measure

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- Flow of the search method is
  - Available values of the preclusion table are accepted as row indices
  - A new preclusion table for the next row index is constructed
  - Drop back to previous row index and table when all row indices have been used in current table
- The search method is conceptually recursive
- A count of the recursion levels entered is a measure of resources required to perform the search

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Symmetry

- {4,2,5,1,3}

	1	2	3	4	5
1				•	
2		•			
3					•
4	•				
5			•		

- {4,2,3,5,1}

	1	2	3	4	5
1					•
2		•			
3			•		
4	•				
5				•	

Sets of Four or Eight

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- Costas arrays formed from other Costas arrays by rotation and transposition are defined as being in a set
- We call Costas arrays of the same set are polymorphs of each other
- Symmetrical Costas arrays belong to sets of four
- Non-symmetrical Costas arrays belong to sets of eight

Full Exploitation of Symmetry

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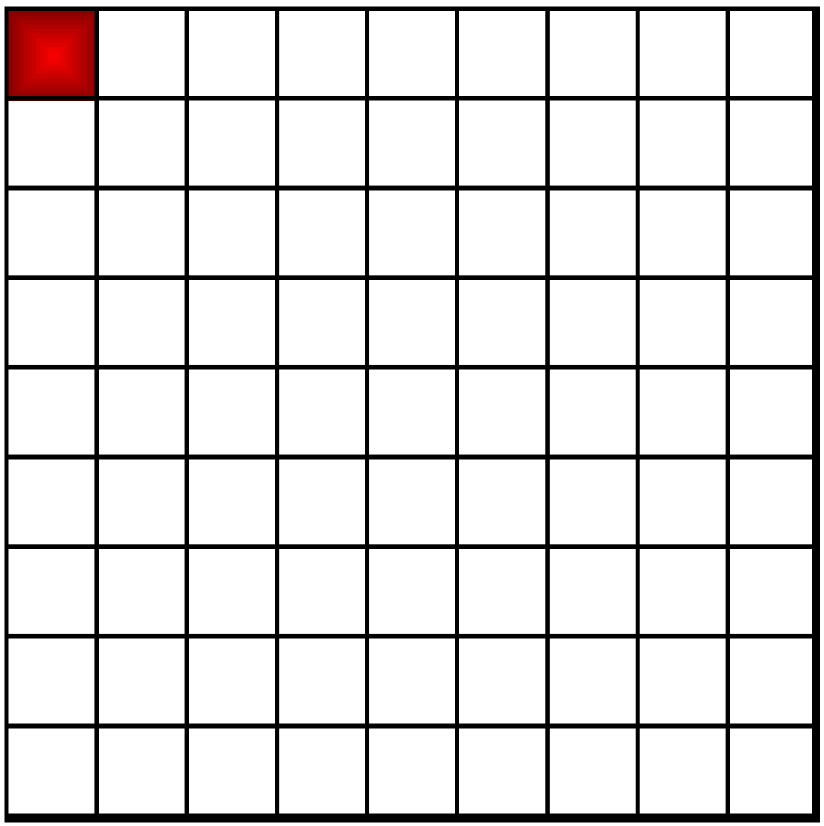
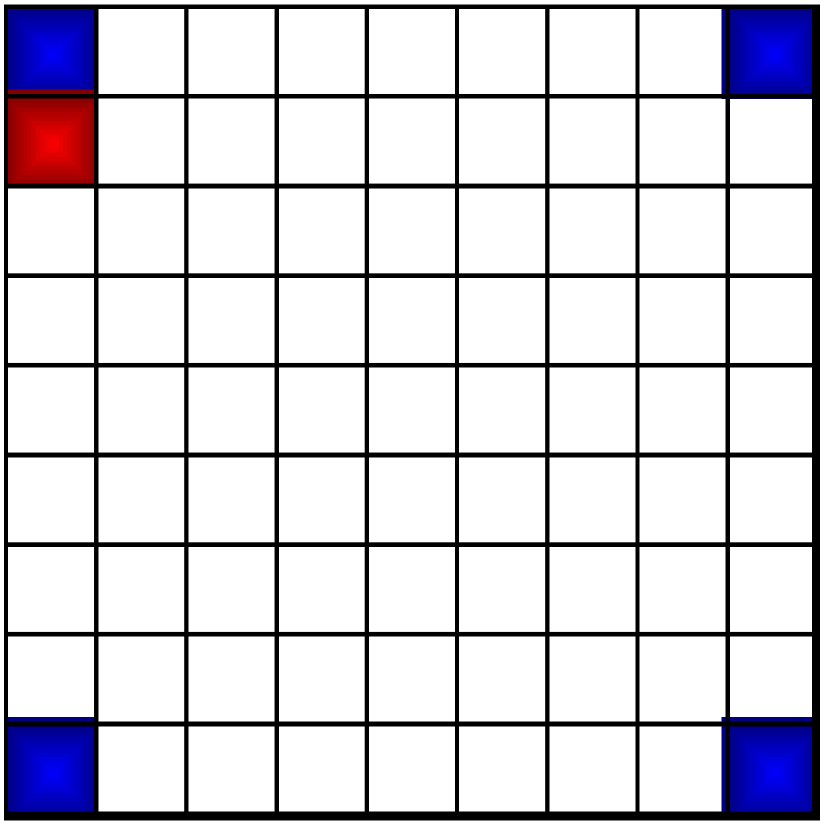


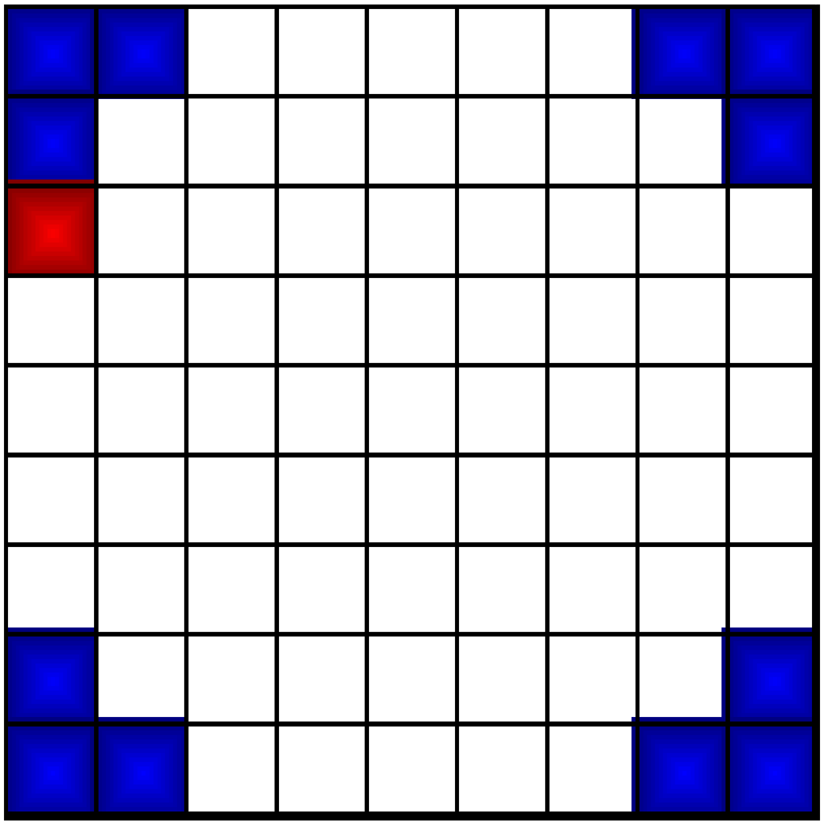
- Search starts with dot in upper left-hand corner
- Finds all possible arrays with this initial dot fixed
- Can be accomplished by corner-dot extending arrays of order N-1
- Omission of this case offers some gains

Exploitation of Symmetry (2)



- Search continues with dot in position (1,2)
- All corner dots may be eliminated from search
- All sets of Costas arrays with a corner dot have already been found

Exploitation of Symmetry (3)



- Continue with dot in position (3,1)
- Corner dots and dots one away from corner are eliminated from search
- Implement by modifying initialization of the preclusion mask

Exploitation of Symmetry (4)

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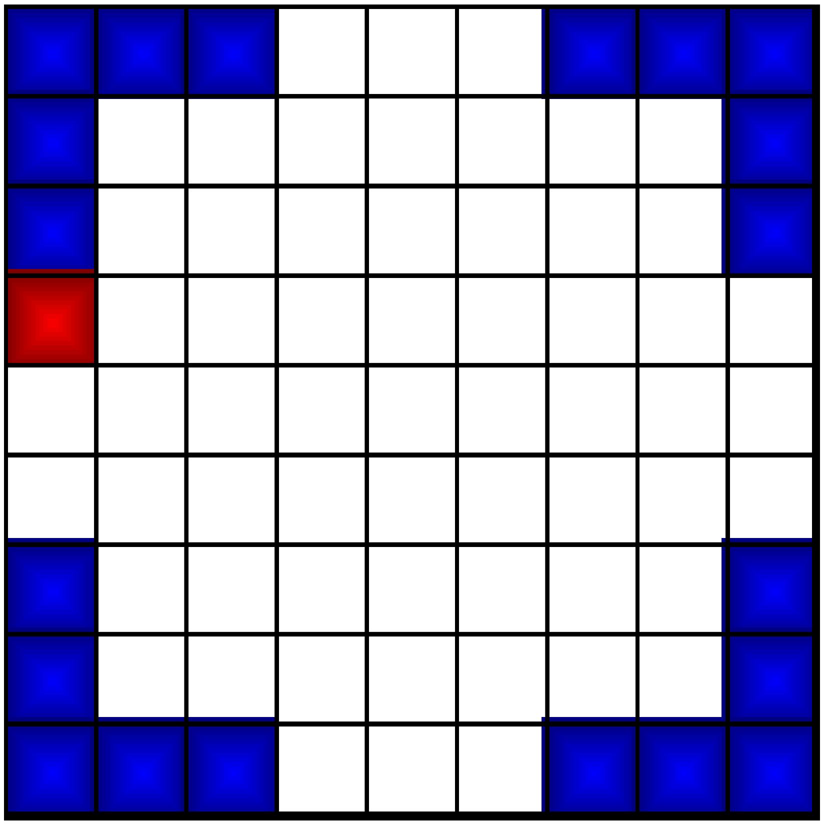

- Continue with dot in position (4,1)
- All edge dots 0 through 2 away from a corner are eliminated
- In general, edge dots closer to any corner than the current fixed dot are eliminated
- They are guaranteed to have been covered by previous row-1 dot position searches

Exploitation of Symmetry (5)

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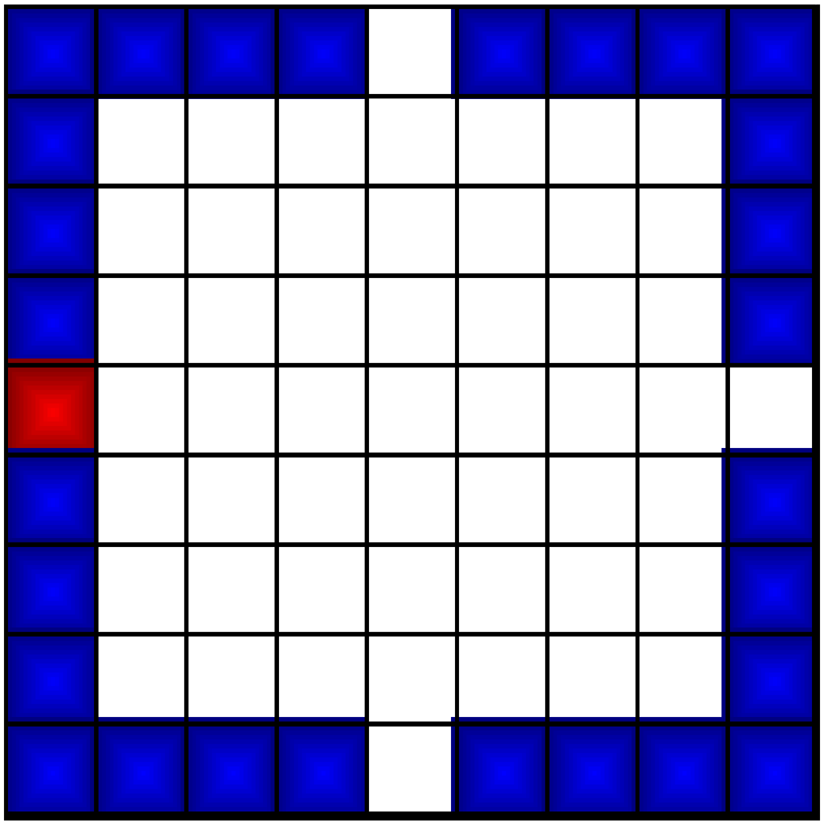

- Center dot not necessary
- There is no place for a[N-1]
- The range of a[0] from 1 to [(N-3)/2] is commonly used

Using More Look-Ahead Levels in Preclusion

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- A given recursion level terminates when its preclusion table is exhausted or filled
- The preclusion table is also computed for the next row index, which is checked for no available row indices
- The preclusion table can be computed for the row index after next, too
- We compute preclusion tables to the last available row

Recursion Counts

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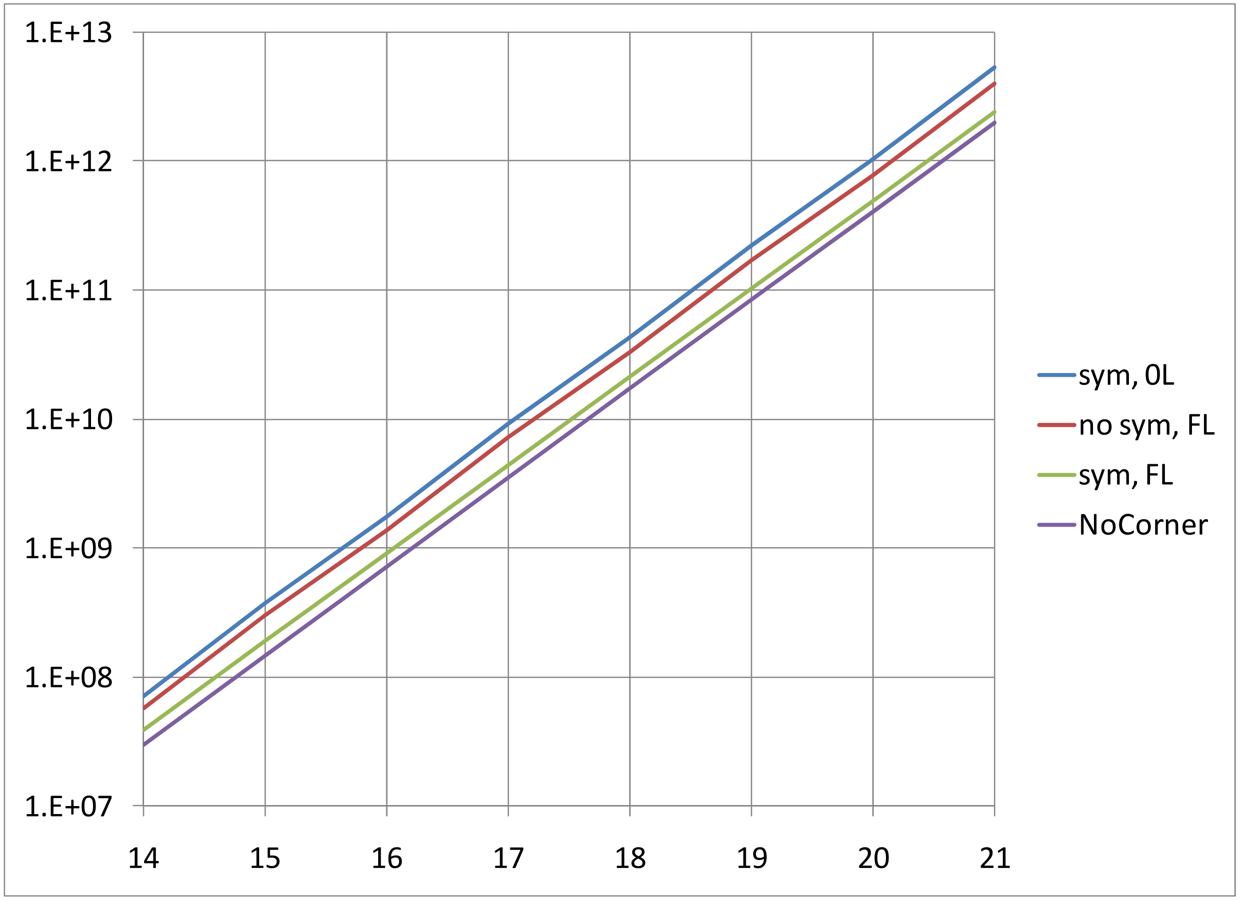
Recursion Counts versus Order

With and Without Symmetry Exploitation, Full Look-Ahead

Counts include Starting with Corner dot Except Lowest Plot



How Does Symmetry Exploitation Do the Job?

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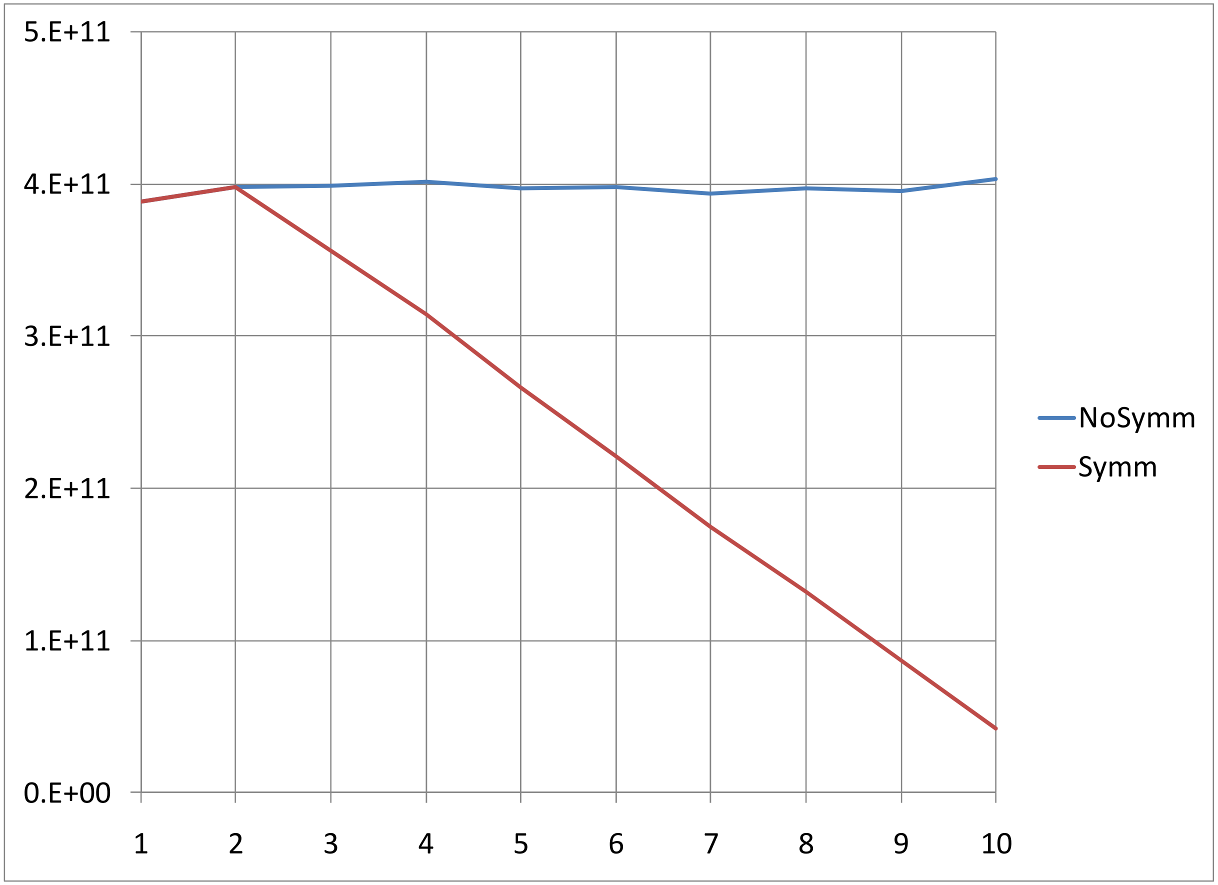
Recursion Counts Versus Row 1 Dot Position for Order 21

Abscissa: a[1] is corner dot

Overall factor of 0.45



How Does Look-Ahead Do The Job?

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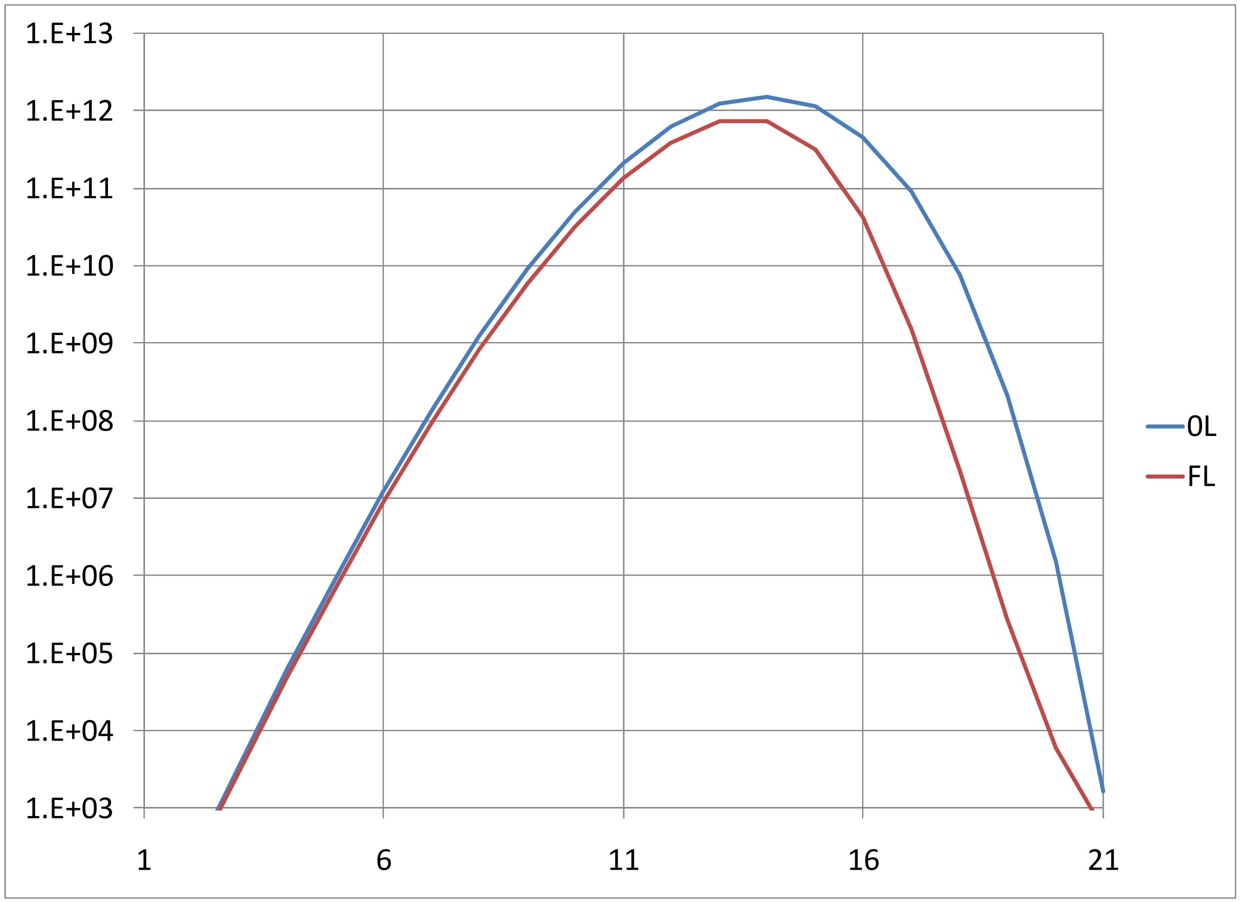
Recursion Counts Versus Recursion Level For Order 21

Overall factor of 0.60



What Can We Expect for Higher Orders?

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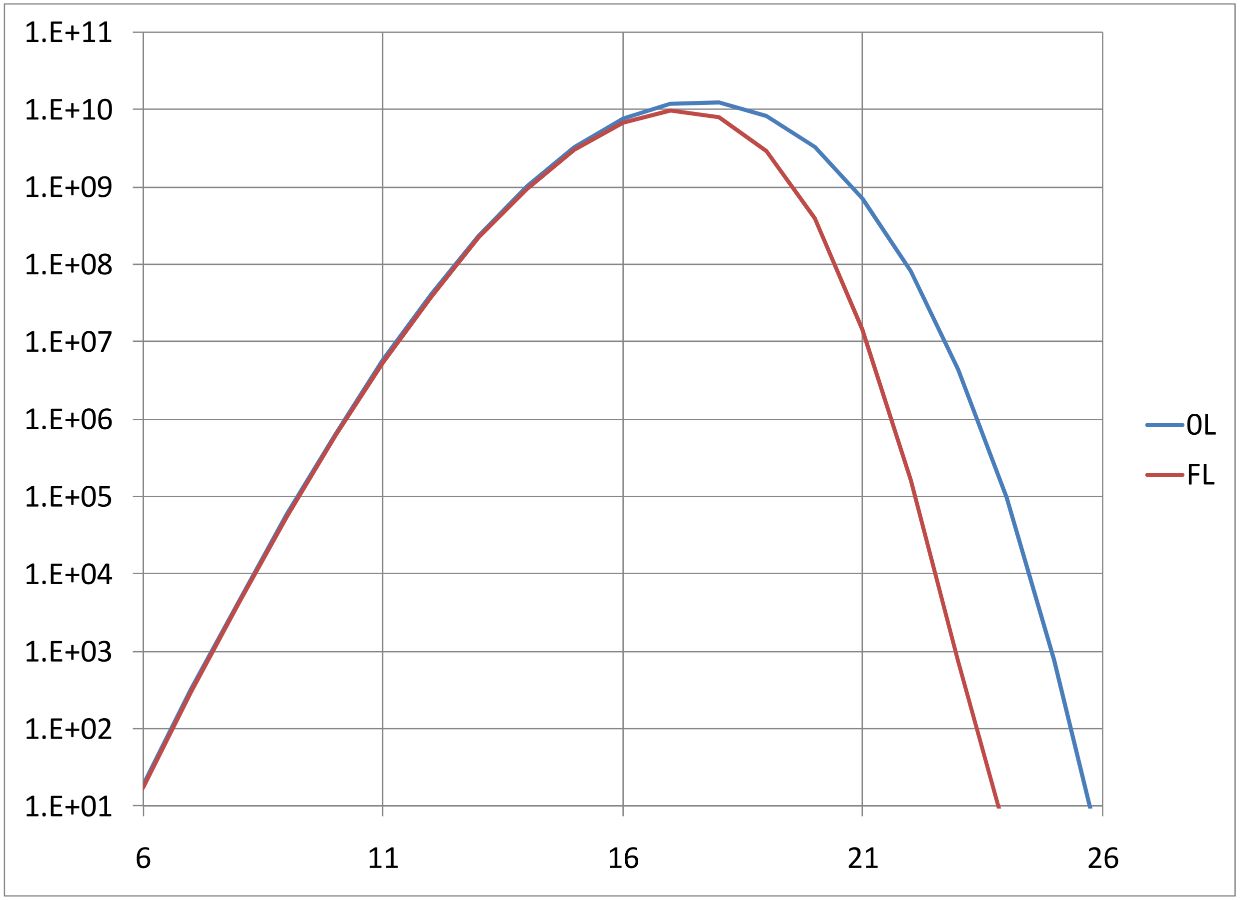
Recursion Counts Versus Recursion Level For Order 28

Search only over {6,26,23,13,15…}



What Gains Are Obtained?

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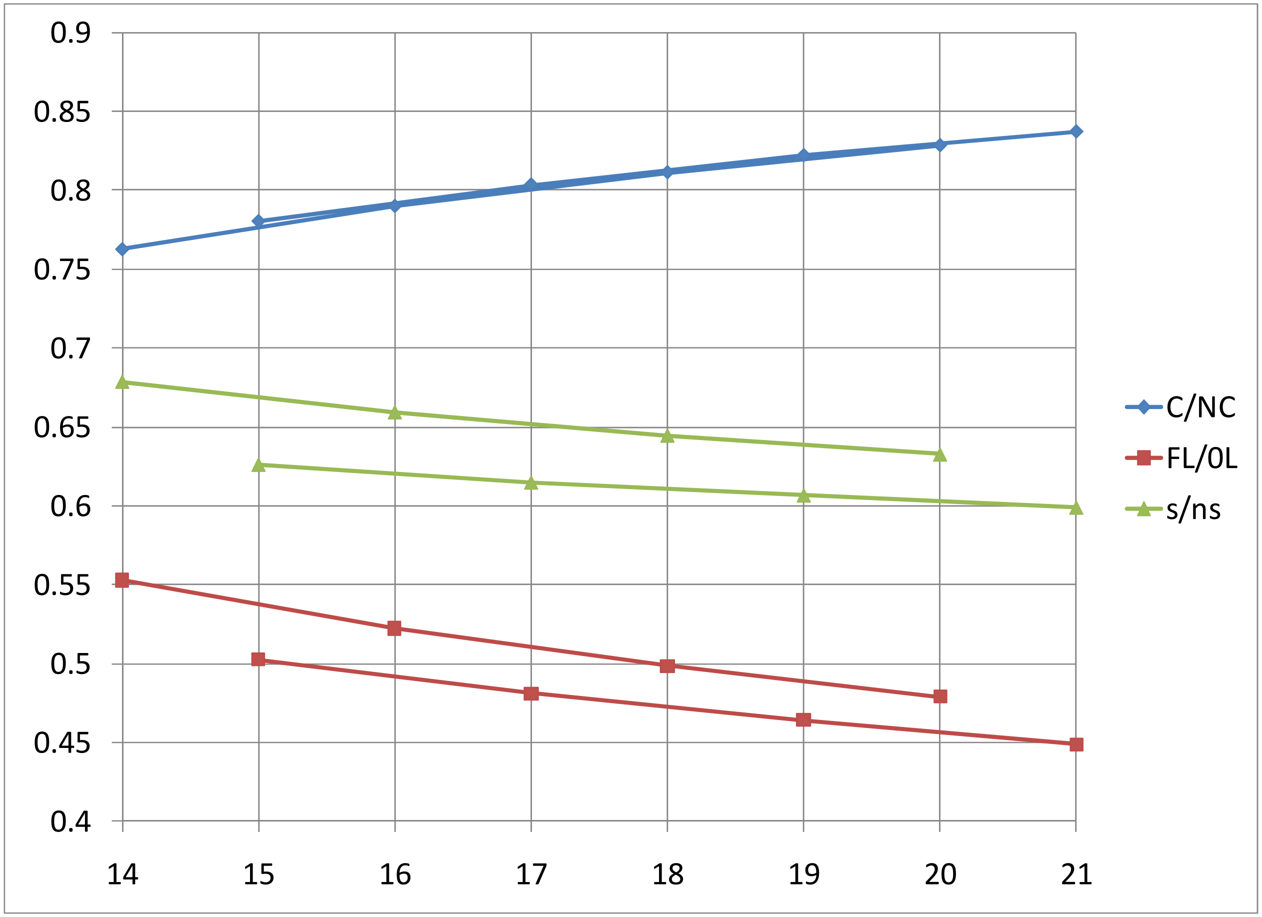
Gains in Recursion Counts vs. Order

Note that odd and even orders are plotted separately



Analyzing the Curves for Orders 14 through 21

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- Least-squares fit, RMS errors 0.0005 or smaller
- Look-Ahead Gains
  - For even order

$$f_{LE}\left(N\right)=0.3074+\frac{3.437}{N},\quad f_{LE}(28)=0.43$$

  - For odd order

$$f_{LO}\left(N\right)=0.3161+\frac{2.8}{N},\quad f_{LO}\left(28\right)=0.42$$

- Symmetry exploitation gains
  - For even order

$$f_{SE}\left(N\right)=0.5253+\frac{2.147}{N},\quad f_{SE}\left(28\right)=0.60$$

  - For odd order

$$f_{SO}\left(N\right)=0.5319+\frac{1.415}{N},\quad f_{SO}\left(28\right)=0.58$$

- Omission of corner dot

$$f_{C}\left(N\right)=0.98-\frac{3}{N},\quad f_{C}\left(28\right)=0.872$$

Observed Look-Ahead Gains at Order 28

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Case	Gain Ratio
{0,5,25,10,26…}	0.71
{1,6,9,5,20…}	0.73
{6,26,23,13,15…}	0.66
(10,22,26,8,2…}	0.31

Overall Gain Ratio of About 0.43 is Reasonable

Total Gains for Order 28

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- From Look-Ahead
  - Extrapolation from curves for orders 14 – 21 estimates a factor of 0.43
  - Corroborated by short sections for actual searches
  - About 2:1 is a very conservative estimate
- From Symmetry exploitation
  - Extrapolation from curves for orders 14 – 21 estimates a factor of 0.60
  - Measured at 0.55 for orders 27 and 28
  - Factor of 0.872 from omission of corner dot
  - On top of 2:1 conventionally obtained by limiting search range of a[0] from 0 to [(N-1)/2]
- TOTAL
  - 0.60 X 0.43 X 0.872 = 0.225, not including conventionally obtained 2:1
  - Some gains from omitting corner dot

A Look at Symmetry Filtering

- All Costas arrays, orders three through 400
  - Contour plot, occurrence versus *a[0]*
  - What full symmetry exploitation accomplishes
- Zoom in to look at orders 3 through 28
- A look at “spurious” Costas arrays

All Costas Arrays

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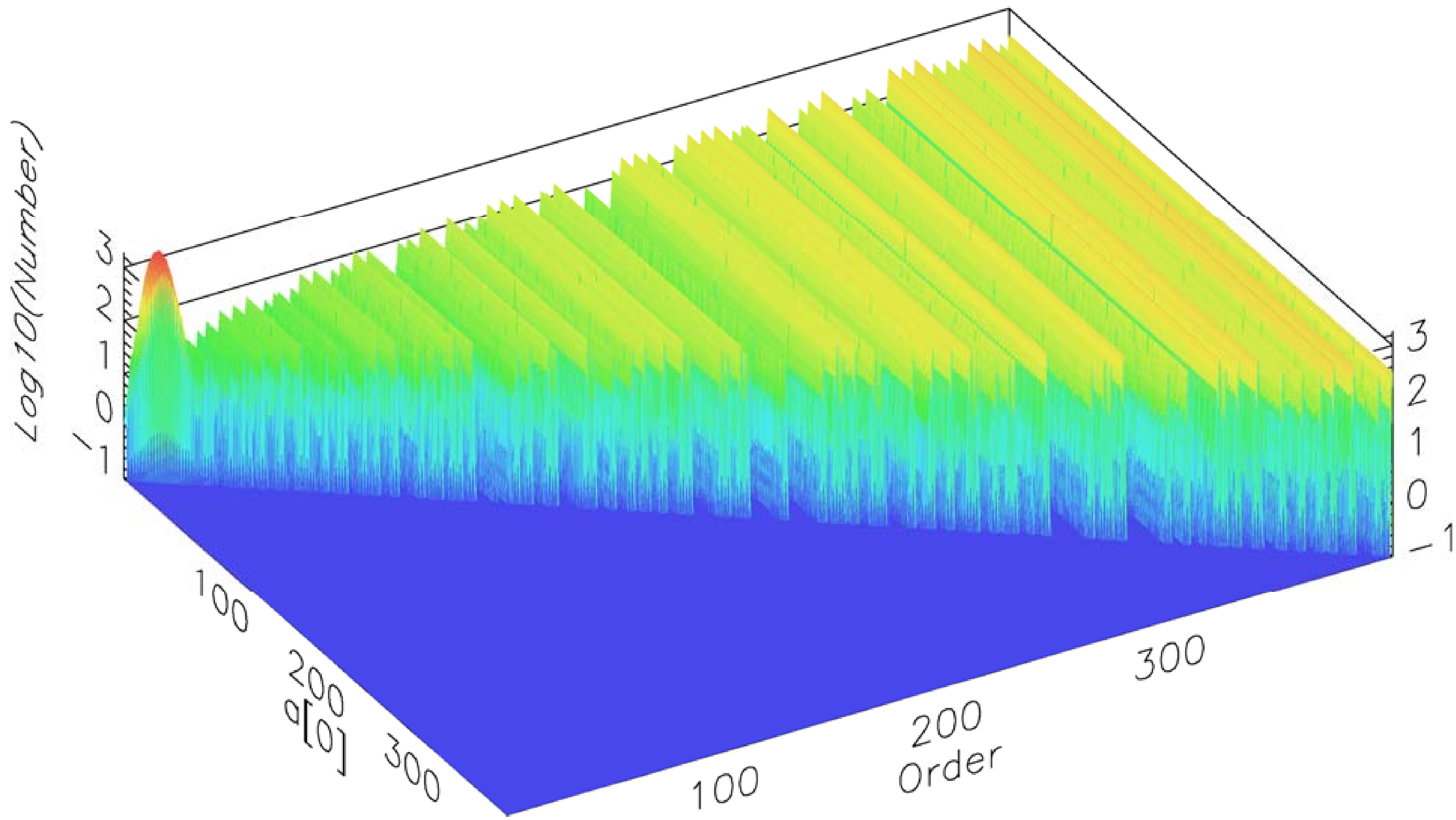

Symmetry Filtered Costas Arrays

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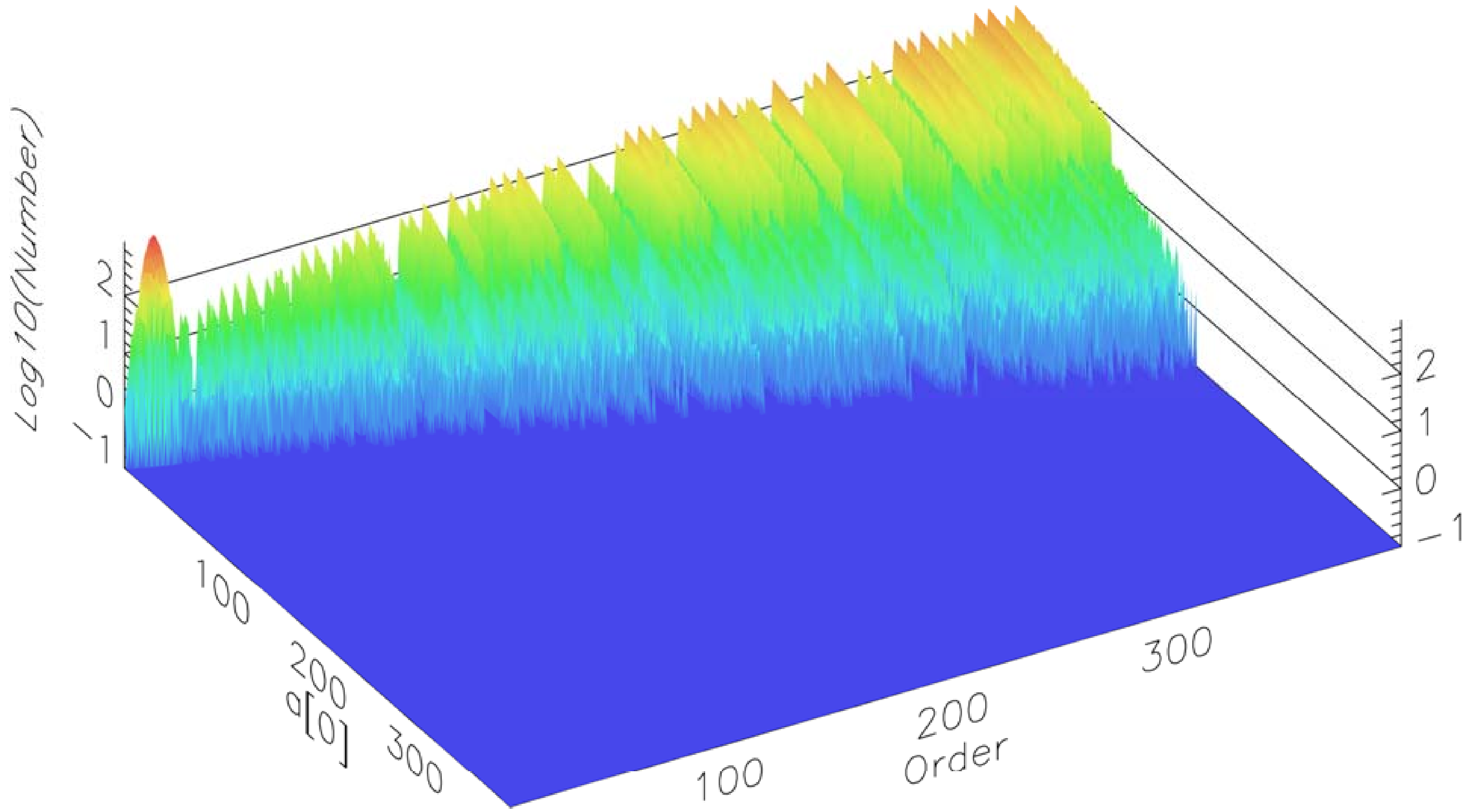

All Costas Arrays

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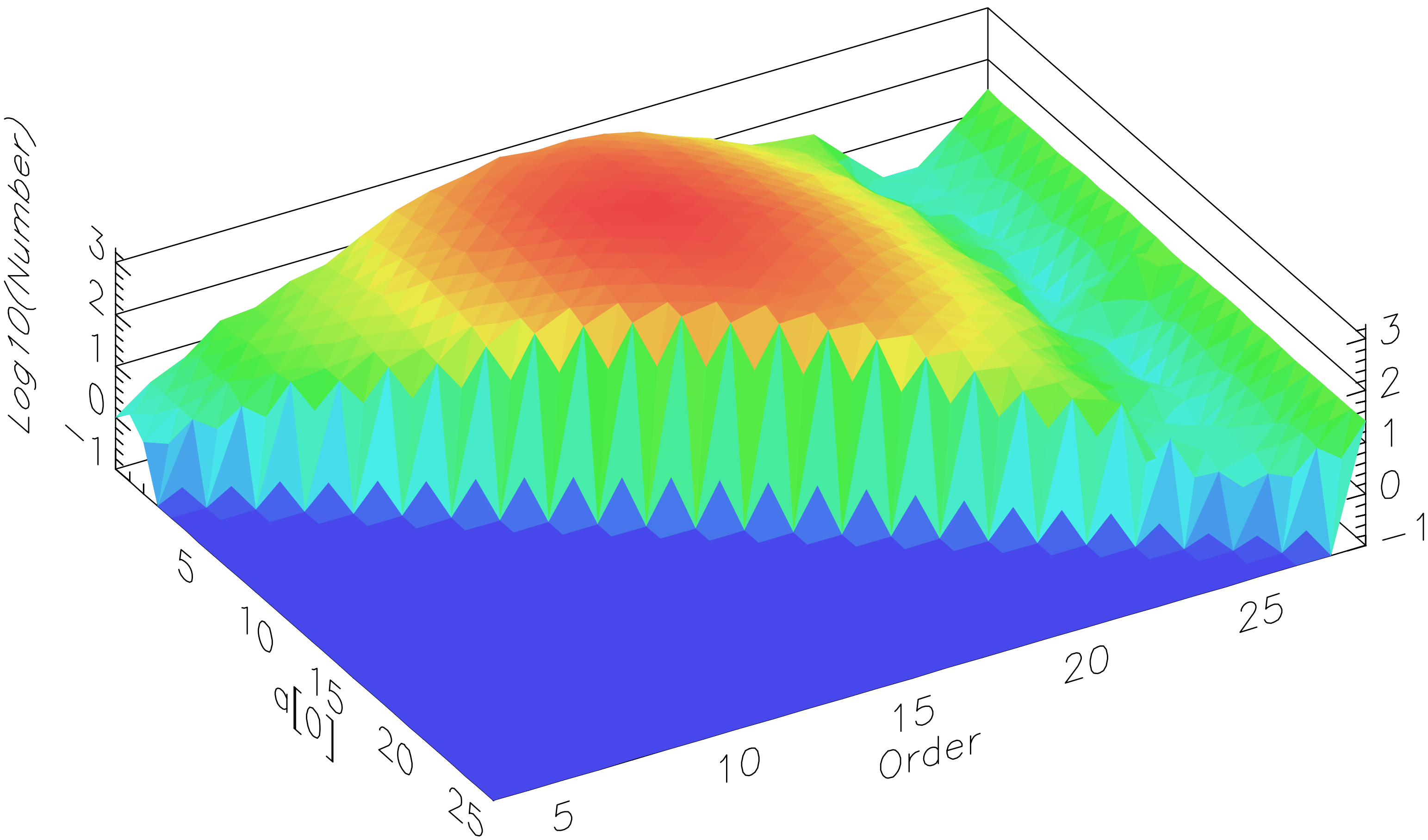

Symmetry Filtered Costas Arrays

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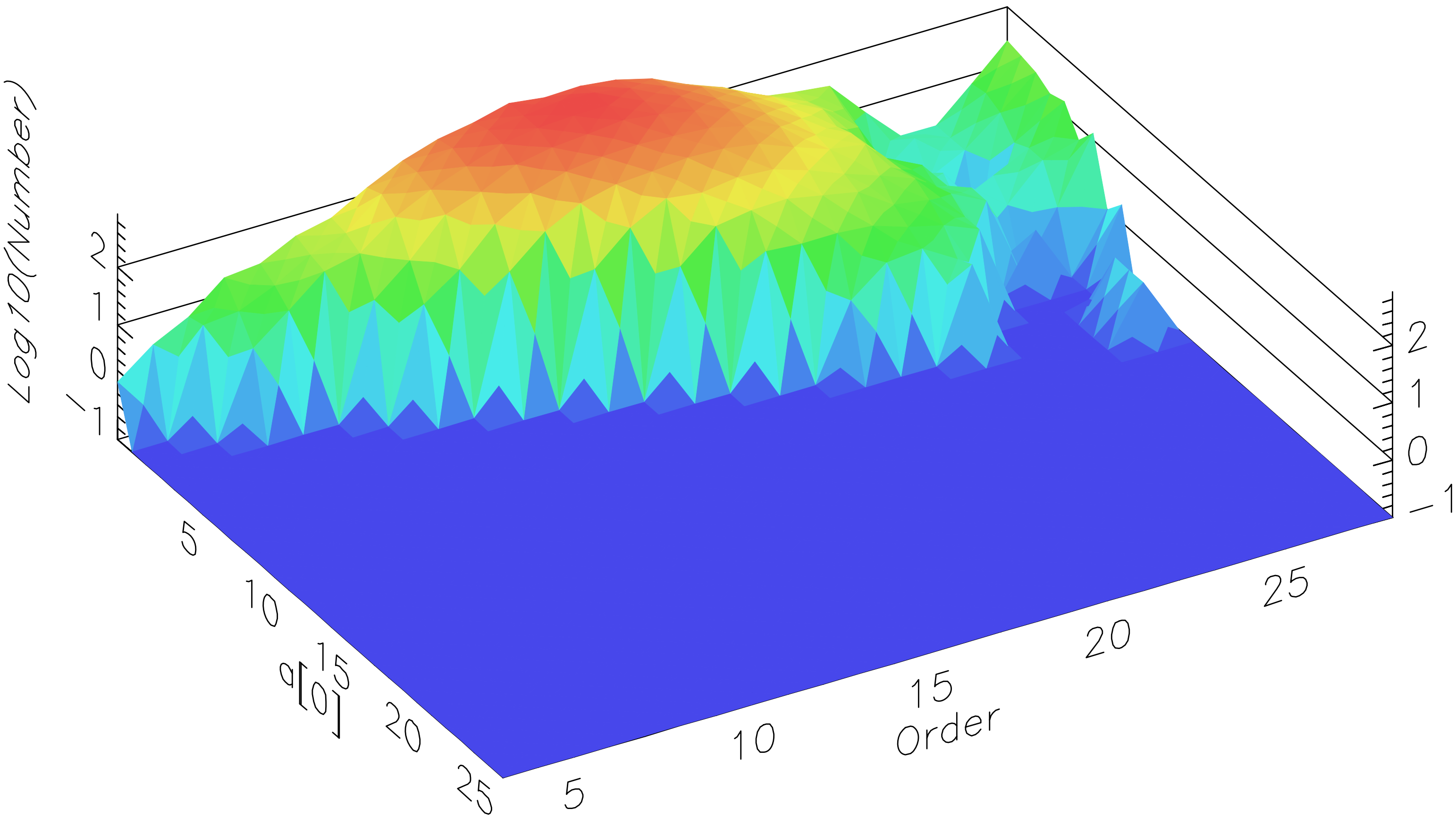
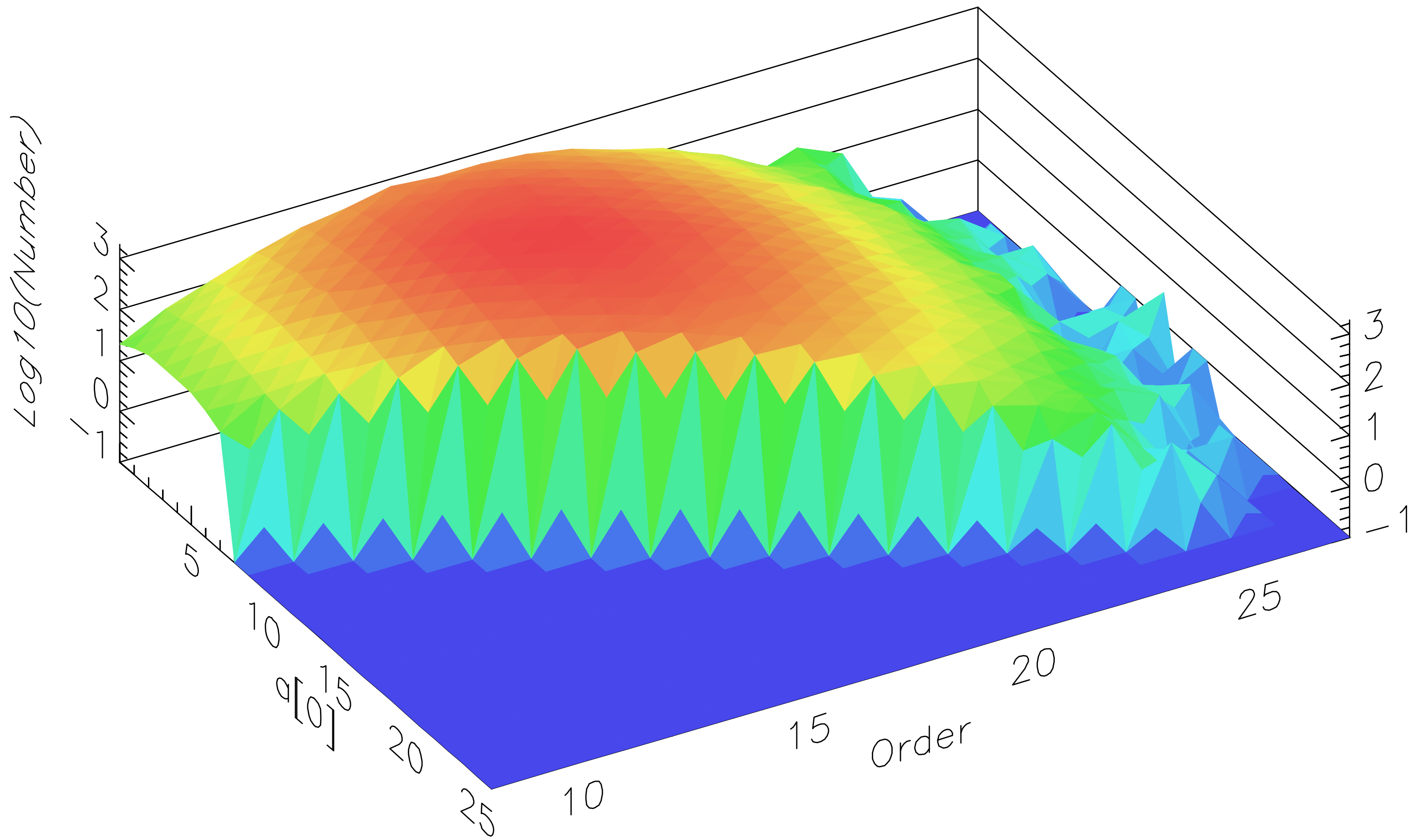


Spurious Costas Arrays



Symmetry Filtered Spurious Costas Arrays

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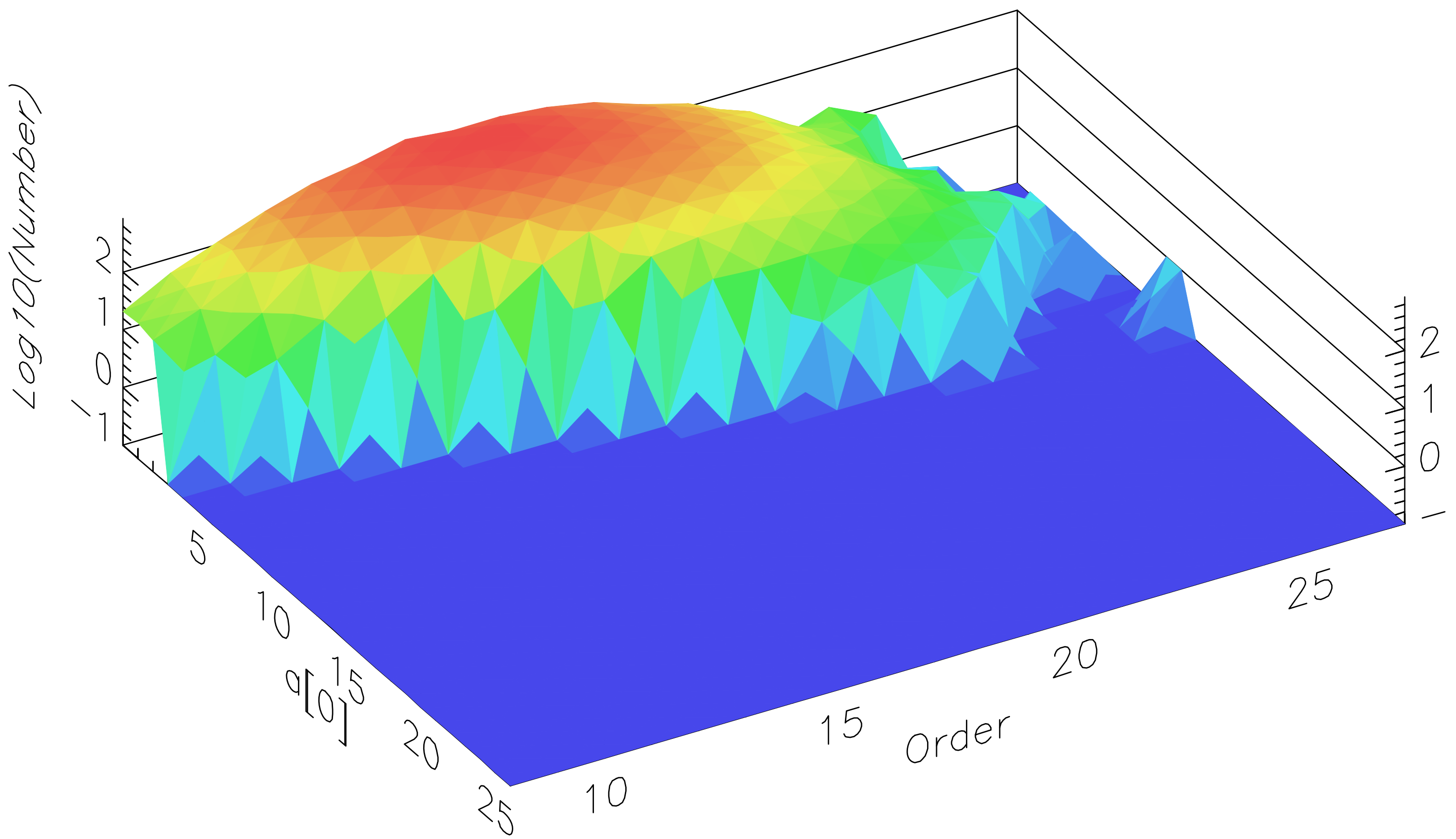

Costas Arrays Found by Search

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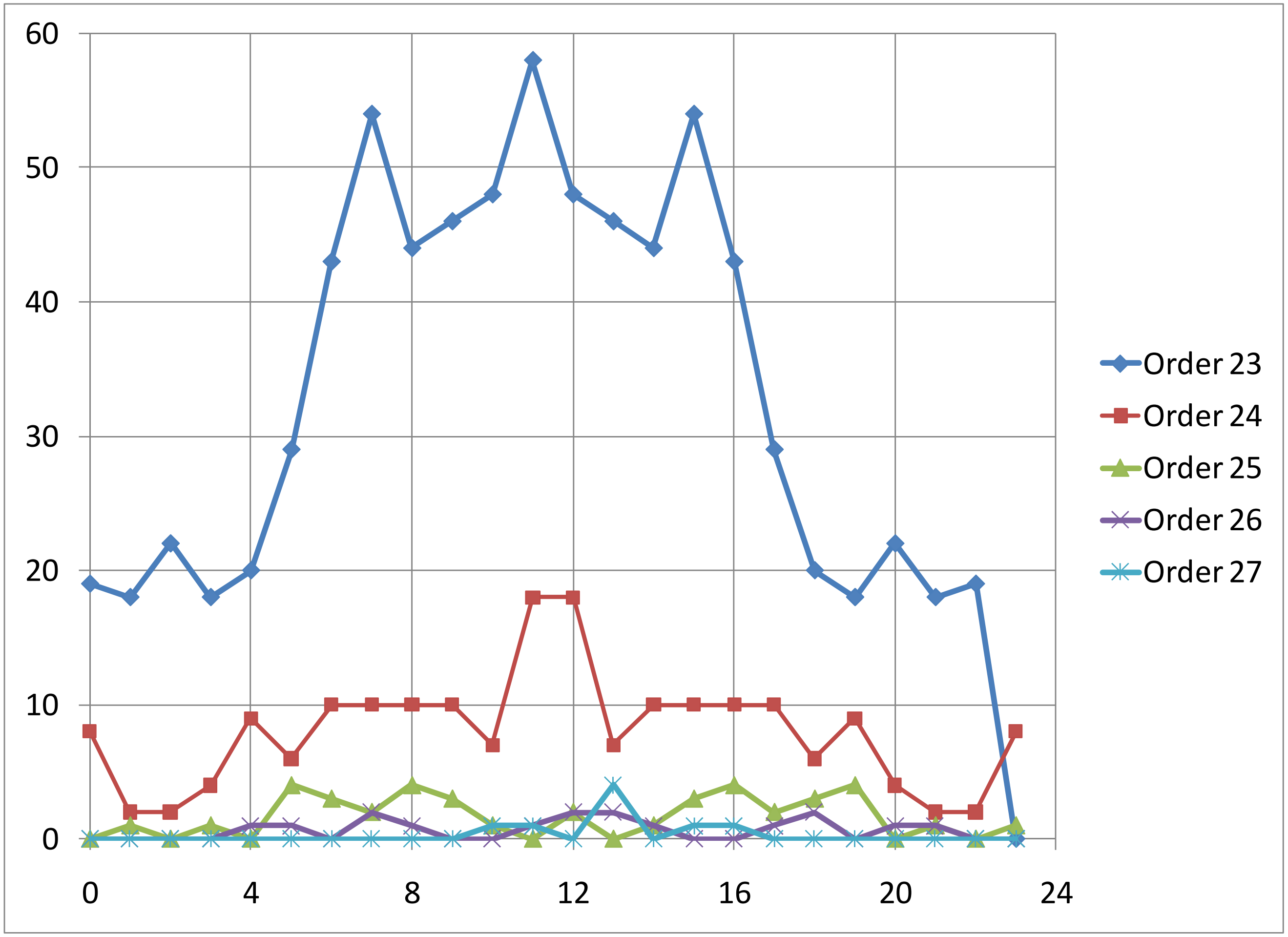
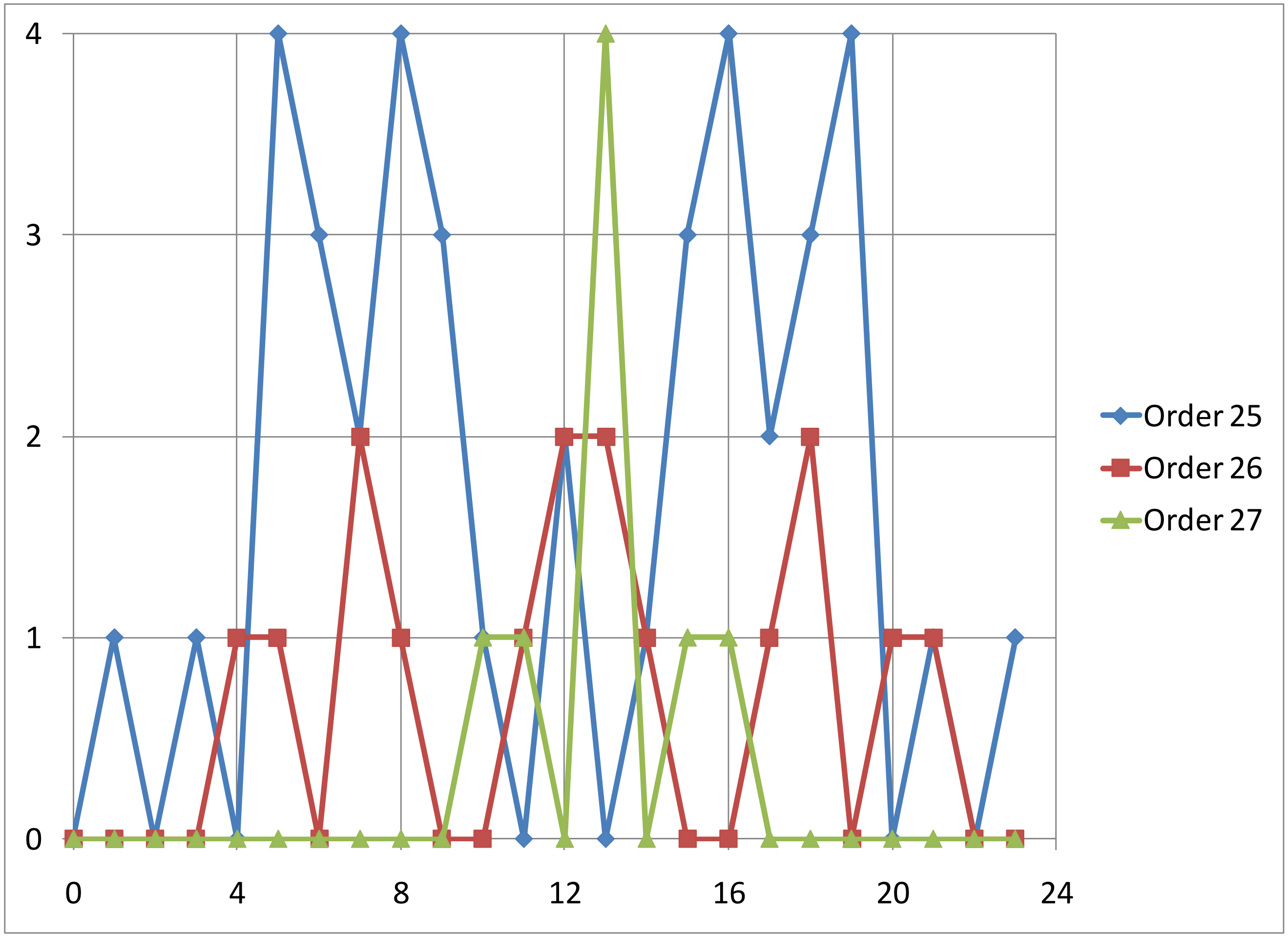
Number Found vs. a[0]



Costas Arrays Found by Search

Number Found vs. a[0]



Costas Arrays Found by Search, Symmetry Filtered

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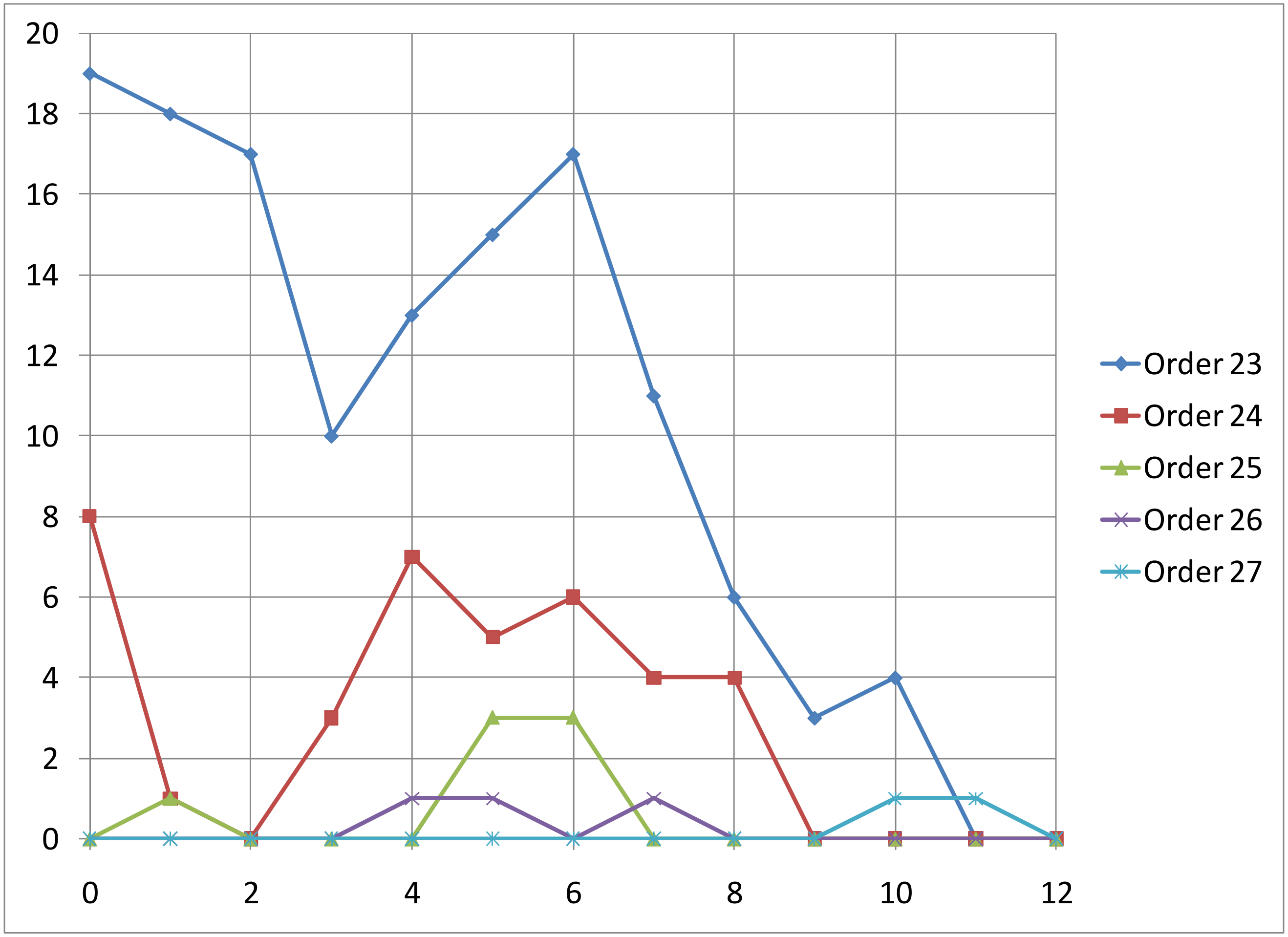
Number Found vs. a[0]



Costas Arrays Found by Search, Symmetry Filtered

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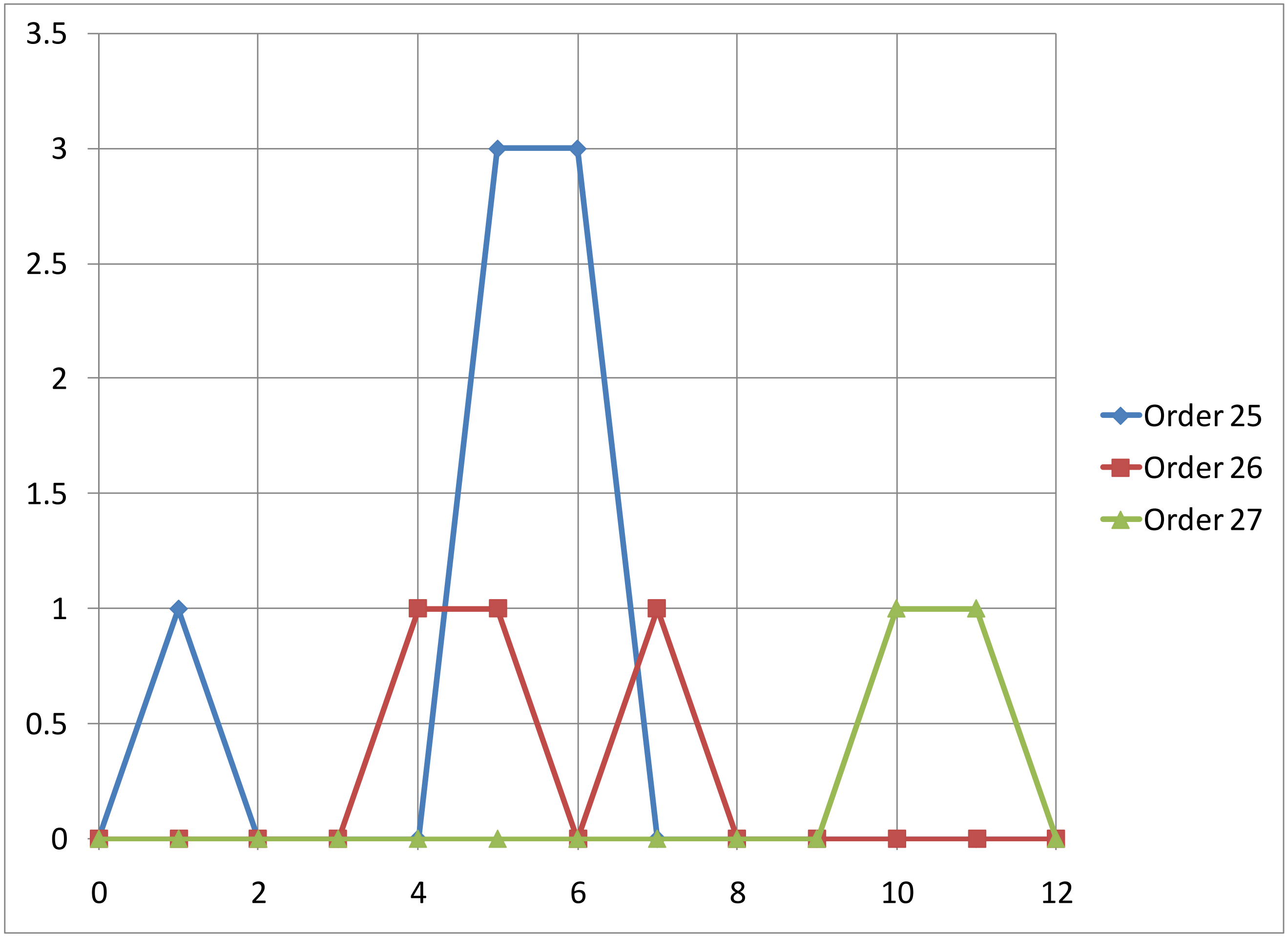
Number Found vs. a[0]



The Last Costas Array

- Found by two teams in Spring of 2008
  - Drakakis, et al., in a log dated March 9 (Euro supercomputer)
  - Beard, et al., found April 8, announced May 29 (personal resources and resources of opportunity)
- Keith G Erickson
  - Identified and accessed resources of opportunity
  - Revised allocations to deal with unique restrictions on the use of this resource
  - Executed allocated searches and found the Costas Array

The Last Costas Array

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- Costas array of order 27
- Few others of order over 26 have ever been found
- Likelihood that any others exist is slight

11	10	4	24	7	23	3	18	21	9	26	16	5	1	15	27	2	25	17	22	19	6	8	12	20	13	14
12	17	10	24	22	8	19	3	7	20	9	16	13	1	2	4	27	26	18	5	23	6	15	25	21	11	14
14	11	21	25	15	6	23	5	18	26	27	4	2	1	13	16	9	20	7	3	19	8	22	24	10	17	12
14	13	20	12	8	6	19	22	17	25	2	27	15	1	5	16	26	9	21	18	3	23	7	24	4	10	11
14	15	8	16	20	22	9	6	11	3	26	1	13	27	23	12	2	19	7	10	25	5	21	4	24	18	17
14	17	7	3	13	22	5	23	10	2	1	24	26	27	15	12	19	8	21	25	9	20	6	4	18	11	16
16	11	18	4	6	20	9	25	21	8	19	12	15	27	26	24	1	2	10	23	5	22	13	3	7	17	14
17	18	24	4	21	5	25	10	7	19	2	12	23	27	13	1	26	3	11	6	9	22	20	16	8	15	14

How Was It Found?

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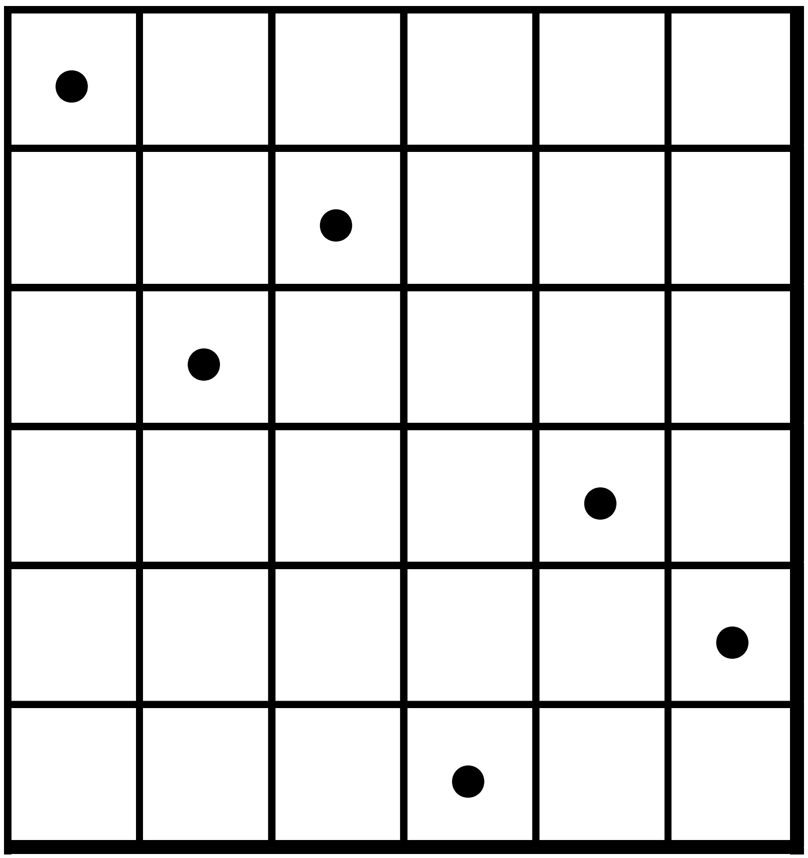
- Multiple independent searches over allocated task space produces mountains of data
- Central bookkeeping methodology
  - Read all the data, every time
  - Produce counts, breakdowns as output
  - Begin with output of extended generator program
  - A change in the count indicates that a new Costas array has been found
  - Comparison of outputs with output from generator program reveals which ones are new
- High-powered CS engines for sort and other tasks allows processing “mountains of data” for data summaries as often as desired

Other Methods

- Augmentation
  - Construct augmented matrix from two Costas arrays
  - Result must satisfy Costas condition
  - Interaction between matrices will almost always result in a violation of the Costas condition
- Interleaving
  - Two Costas arrays with orders differing by at most one
  - Construct checkerboard interleaved matrix

A Remarkable Example



Cumulative Totals versus Order

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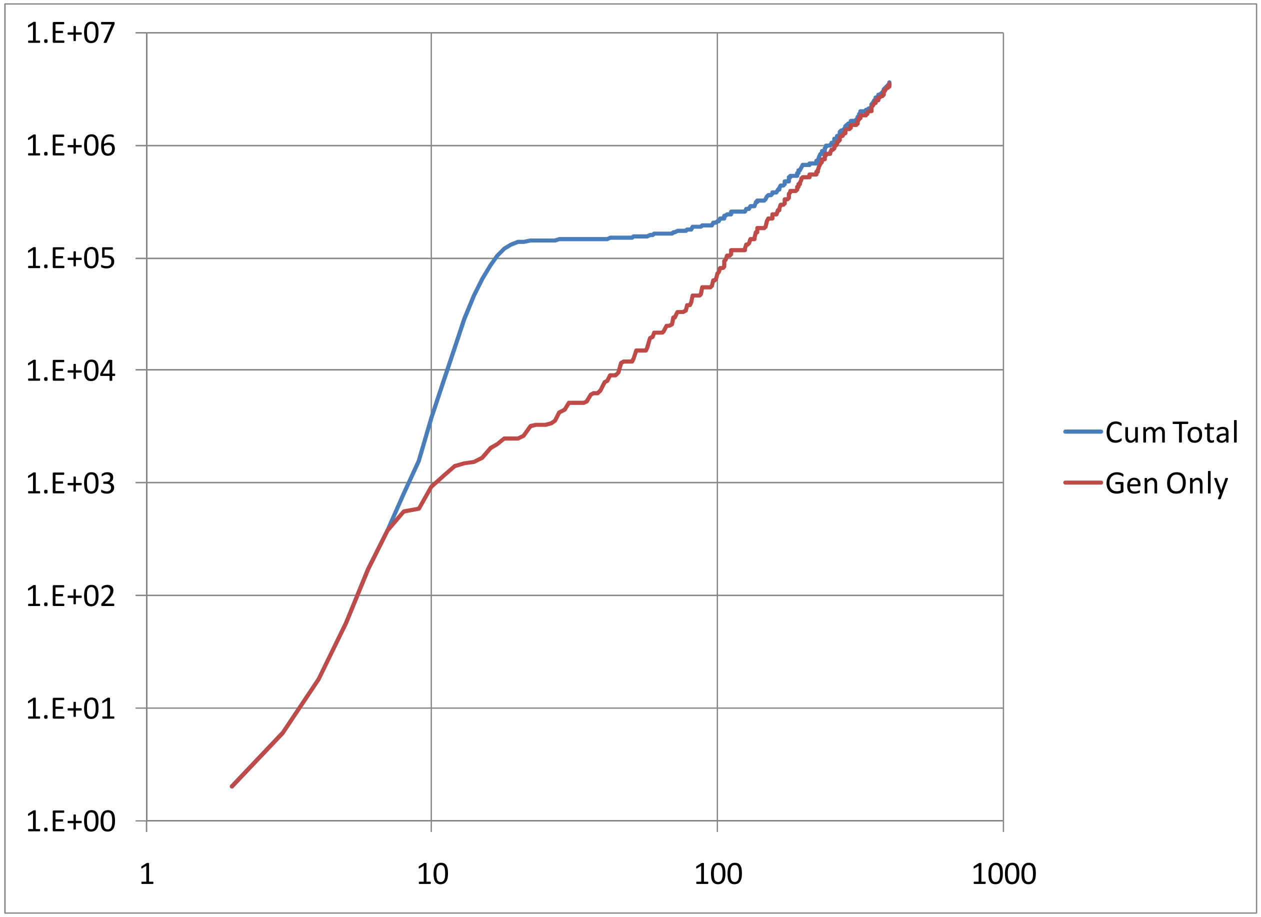

Summary

- Search algorithm developed by this team
- Computational resources were unremarkable
  - Desktop computers owned and maintained by team
  - Occasional off-hours use of desktops by consenting organizations
  - Other temporary resources of opportunity as identified and exploited by team members
- Published first completed searches over orders 24, 25, 26
- Found and first reported last Costas array of order 27

Conjectures (1 of 2)

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- No Costas arrays above order six exist that have two empty quadrants.
- Orders 32 and 33 will be searched within the next 15 years. No Costas arrays of those orders will be found.
- The current generalizations and generators find all that exist above order 27.
- The number of Costas arrays of any given order $N>23$ does not exceed N^2. [FALSE]
- Cumulative count fits $0.19 \cdot N^{2.78}$ for large N

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Conjectures (2 of 2)

- The number of consecutive orders *K* for which no Costas arrays exist has no upper bound. But, for any order *L*, an order *N* exists for which there are Costas arrays, and max{|*L/N-1*|} has no lower bound as *L* increases.
- The value of Costas arrays in spectrum sharing will make them ubiquitous in communications and radar waveforms.
- The 2-D correlation properties of Costas arrays will make them fundamental to digital fingerprinting.

Acknowledgements

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- Wright and Monteleone participated through orders 26
- LMCO ATL Publications proofread and reviewed this work
- Keith G Erickson's contributions include the discovery of "the last Costas array"
  - Identified unique resources of opportunity
  - Defined revision of allocation to condition data to meet restrictions of this resource
  - Exploited this resource and produced results to Team Bookkeeping central processing

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