

Contents

Dedication	v
Author	xiii
Preface	xv
Foreword	xvii
The Fourier Transform	1
1. Overview	1
2. Conventions and Notations	1
2.1 COMPLEX VARIABLES AND THE COMPLEX CONJUGATE	2
2.2 VECTORS AND MATRICES	2
2.3 MATRIX TRANSPOSE AND HERMITIAN	3
2.4 COMMON FUNCTIONS	3
3. The Fourier Transform	4
3.1 THE CLASSICAL FOURIER TRANSFORM	4
3.2 OUR FIRST ENCOUNTER WITH THE DIRAC DELTA FUNCTION	4
3.3 MEANING, USEFULNESS, AND LIMITATIONS OF FORMAL IDENTITIES	6
3.4 THE INVERSE FOURIER TRANSFORM	6
3.5 PARSEVAL'S THEOREM	7
3.6 MULTIVARIATE FOURIER TRANSFORMS	8

3.6.1	Two-Dimensional Fourier Transforms	8
3.6.2	Three and More Dimensions	10
3.7	FOURIER TRANSFORM PAIRS	13
3.8	THE HILBERT TRANSFORM	14
4.	The Classical Fourier series	16
4.1	THE FOURIER SERIES	16
4.2	PARSEVAL'S THEOREM	17
5.	The Discrete Fourier Transform	17
5.1	THE TRANSFORM – A TRIGONOMETRIC IDENTITY	17
5.2	PARSEVAL'S THEOREM	18
5.3	THE DIRICHLET KERNEL	18
5.4	DFT PAIRS	19
6.	Gibb's Phenomenon	20
7.	Spatial and Matrix Representations and Interpretations	22
7.1	THE CONTINUOUS FOURIER TRANSFORM	22
7.1.1	In Engineering and Physics	22
7.1.2	In Mathematics	24
7.2	THE CLASSICAL FOURIER SERIES	25
7.3	THE DISCRETE FOURIER TRANSFORM	26
7.3.1	Filtering and Inverse Transforming	26
7.3.2	Properties of the DFT and Fourier Matrices	27
8.	Problems	30
8.1	GENERAL	30
8.2	CLASSICAL FOURIER TRANSFORM	30
8.3	CLASSICAL FOURIER SERIES	31
8.4	DISCRETE FOURIER TRANSFORM	31
8.5	GREATER TIME AND DIFFICULTY	31
8.6	PROJECT	31
	Introduction to the Radix 2 FFT	33
1.	Historical Note	33
2.	Notations and Conventions	33
3.	Ordering the Bits in the Addresses	34
4.	Examples	35

4.1	SIMPLE DIF AND DIT	35
4.2	MULTIVARIATE FFT	38
5.	Problems	50
5.1	GENERAL	50
5.2	GREATER DIFFICULTY	51
5.3	PROJECT	51
The Reordering Problem and its Solutions		53
1.	Introduction	53
2.	Different types of Cooley-Tukey FFTs	55
3.	In-place, self reordering FFTs	56
4.	Conclusions	60
4.1	SUMMARY	60
4.2	EXECUTION SPEEDS	61
4.3	VARIABLE RADIX ALGORITHMS	62
4.4	MULTIVARIATE FFTS	62
5.	Examples	63
6.	Problems	80
6.1	GENERAL	80
6.2	GREATER DIFFICULTY	81
6.3	PROJECT	81
Spectral Window Weightings		83
1.	Overview	83
1.1	BASE CONCEPTS – THE DFT TRADE SPACE	83
1.2	CONTINUOUS AND DISCRETE SPECTRAL WINDOWS	85
1.3	SAMPLED CONTINUOUS SPECTRAL WINDOWS	85
1.4	NOISE BANDWIDTH	86
1.5	ARRAY EFFICIENCY	87
1.6	SPECTRAL WINDOW FREQUENCY RESPONSE	87
2.	Discrete Spectral Windows	89
2.1	THE DOLPH-CHEBYCHEV WINDOW	89
2.2	CHEBYCHEV 2 WINDOW	98
2.3	SPLIT CHEBYCHEV 2 WINDOW FOR MONOPULSE	107
2.4	FINITE IMPULSE RESPONSE FILTERS	112

3.	Continuous Spectral Windows and Sampled Continuous Windows	113
3.1	SAMPLED CONTINUOUS WINDOW FUNCTIONS AS DISCRETE WINDOWS	113
3.2	COSINE WINDOWS	113
3.2.1	Bartlett and Hanning	113
3.2.2	Placing Zeros: Hamming, Blackman, and Harris Windows	114
3.2.3	A Fifth Cosine for 118 dB Performance	116
3.2.4	Cosine to a Power	130
3.3	CONTINUOUS EXTENSIONS OF THE DOLPH-CHEBYCHEV WINDOW	131
3.3.1	Base Continuous Dolph-Chebychev Window	131
3.3.2	The Kaiser Window	141
3.3.3	The Taylor Window	146
4.	Two-Dimensional Window Weightings	159
4.1	PLANAR RADAR ANTENNAS AND TWO-DIMENSIONAL DFTS	159
4.2	THE TWO-DIMENSIONAL DOLPH-CHEBYCHEV WEIGHTING	162
4.3	TWO-DIMENSIONAL CHEBYCHEV 2 WINDOW	171
4.4	MONOPULSE WITH SPLIT TWO-DIMENSIONAL CHEBYCHEV 2 WINDOW	178
4.5	LIMITING FORM OF THE TWO-DIMENSIONAL CHEBYCHEV WINDOWS FOR LARGE N	181
4.6	TWO-DIMENSIONAL TAYLOR WEIGHTING	183
4.7	LAMBDA FUNCTIONS AND A UNIFIED THEORY	199
4.8	MONOPULSE WITH THE BAYLISS WINDOW WEIGHTS	200
5.	Three and More Dimensions	203
5.1	CHEBYCHEV	204
6.	Linear Programming Window Function Design	204
7.	Conclusions	205
7.1	ONE-DIMENSIONAL WEIGHTINGS	205
7.2	TWO-DIMENSIONAL WEIGHTINGS	206
7.3	THREE AND HIGHER DIMENSIONS	207
8.	Problems	207
8.1	GENERAL	207
8.2	GREATER DIFFICULTY	207
8.3	CHEBYCHEV WINDOWS	208
8.4	COSINE WINDOWS	208
8.5	BESSEL WINDOWS	209

8.6	TWO-DIMENSIONAL WINDOWS	209
8.7	GREATER TIME AND DIFFICULTY	209
8.8	PROJECT	209
	Acknowledgments	211
	References	213
	Index	219

$$afg_k = \frac{1}{N^2} \cdot \sum_{i=0}^{N-1} \sum_{p=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} af_m \cdot ag_n^* \cdot \exp\left(-j \cdot \frac{2p}{N} \cdot (i \cdot k - p \cdot m + (p-i) \cdot n)\right) \quad (3.56)$$

Summation on i is possible because only complex exponentials are involved, and the result is a Dirichlet kernel, which is effectively a Kronecker delta scaled by N , or $N \cdot \mathbf{d}_{k,n}$. This allows summation on n as well, with the only nonzero term on summation on n being the one for $n = k$. The result is

$$afg_k = \frac{1}{N} \cdot \sum_{p=0}^{N-1} \sum_{m=0}^{N-1} af_m \cdot ag_k^* \cdot \exp\left(-j \cdot \frac{2p}{N} \cdot p \cdot (-m+k)\right). \quad (3.57)$$

We can now sum on p and find another Dirichlet kernel, then sum on m to find

$$afg_k = af_k \cdot ag_k^*. \quad (3.58)$$

These and other results are summarized below as Table 2.

Table 1-2. DFT/FFT Transform Pairs

x_i	$a_i = \sum_{i=0}^{N-1} x_i \cdot \exp\left(-j \cdot \frac{2p}{N} \cdot i \cdot k\right)$	Remarks
a_k	$N \cdot x_{N-k}$	Double transform
$\sum_{p=0}^{N-1} f_p \cdot g_{i-p}^*$	$af_k \cdot ag_k^*$	Convolution, cross-correlation
$\sum_{p=0}^{N-1} f_p \cdot f_{i-p}^*$	$ a_k ^2$	Autocorrelation, Energy spectrum
$f_i \cdot g_i^*$	$\frac{1}{N} \cdot \sum_{p=0}^{N-1} af_p \cdot ag_{p-i}^*$	Multiplication, Convolution
$\mathbf{d}_{i,p}$	$\exp\left(-j \cdot \frac{2p}{N} \cdot p \cdot k\right)$	Kronecker delta
$\exp\left(+j \frac{2p}{N} \cdot i \cdot s\right)$	$\frac{\sin\left(\frac{p}{N} \cdot (k-s)\right)}{\sin\left(\frac{p}{N} \cdot (k-s)\right)} \exp\left(-j \cdot \frac{p \cdot (N-1)}{N} \cdot (k-s)\right)$	Dirichlet kernel

6. GIBB'S PHENOMENON

Gibb's phenomenon is the behavior of an inverse Fourier transform, Fourier series, or DFT near a step discontinuity. For the Fourier transform, we see it when we look at the Fourier transform of the step function. For the Fourier transform, we come upon it looking at the unit step function in time,

Table 1-2. DFT/FFT Transform Pairs

x_i	$a_k = \sum_{i=0}^{N-1} x_i \cdot \exp\left(-j \cdot \frac{2\mathbf{p}}{N} \cdot i \cdot k\right)$	Remarks
a_k	$N \cdot x_{N-k}$	Double transform
$\sum_{p=0}^{N-1} f_p \cdot g_{i-p}^*$	$af_k \cdot ag_k^*$	Convolution, cross-correlation
$\sum_{p=0}^{N-1} f_p \cdot f_{i-p}^*$	$ a_k ^2$	Autocorrelation, Energy spectrum
$f_i \cdot g_i^*$	$\frac{1}{N} \cdot \sum_{p=0}^{N-1} af_p \cdot ag_{p-k}^*$	Multiplication, Convolution
$d_{i,p}$	$\exp\left(-j \cdot \frac{2\mathbf{p}}{N} \cdot p \cdot k\right)$	Kronecker delta
$\exp\left(+j \frac{2\mathbf{p}}{N} \cdot i \cdot s\right)$	$\frac{\sin(\mathbf{p} \cdot (k-s))}{\sin\left(\frac{\mathbf{p}}{N} \cdot (k-s)\right)} \cdot \exp\left(-j \cdot \frac{\mathbf{p} \cdot (N-1)}{N} \cdot (k-s)\right)$	Dirichlet kernel

Note that our normalization of A differs from that of Taylor's paper for consistency with other paragraphs here. The number of "equiripple" sidelobes \bar{n} must be large enough so that the broadening factor \mathbf{S} is greater than one because the factor of \mathbf{S} applies to the main lobe, and adding rolloff to the sidelobes cannot decrease main lobe width. Also, \bar{n} must be large enough so that \mathbf{S} decreases with increasing \bar{n} , because increasing the bandwidth over which the sidelobes are equiripple must decrease the main lobe width. This condition,

$$\bar{n} \geq \frac{1}{2} \cdot \left(4 \cdot \left(\frac{A}{\mathbf{p}} \right)^2 + 1 \right) \quad (3.41)$$

is a hard limit on applicability – the frequency response of the Taylor window does resemble its desired shape when this condition is met, and the sidelobe structure is less clearly related to the design intent when this condition is violated. The minimax principle shows that, as Taylor stated in his original paper⁵⁰, the first \bar{n} sidelobes cannot be down as far as designed unless \mathbf{S} is greater than one, which can only be true when Equation (3.41) is satisfied.

We examine \mathbf{S} as a function of \bar{n} , in the abstract. The broadening factor \mathbf{S} is zero for \bar{n} equal to zero, reaches a peak greater than one at the value given as the threshold in Equation (3.41), and decreases to an asymptote of one as \bar{n} increases past that of the threshold value. This peak value is

$$\mathbf{S}_{MAX} = \sqrt{1 + \frac{1}{4 \cdot \left(\frac{A}{\mathbf{p}} \right)^2}} \quad (3.42)$$

which will be near one when A is large, that is to say when S is very large. Furthermore, decrease of \mathbf{S} as \bar{n} increases above the threshold value will be quite slow. Since the sidelobes of the frequency response are observed to be essentially as designed when Equation (3.41) is satisfied, practical designs are obtained by simply rounding up from the threshold value, or perhaps adding one or two to the threshold value before rounding. Increasing \bar{n} much beyond that which is required to obtain the sidelobe heights will provide the frequency response shape according to the theory, but the behavior of the weighting function will begin to show artifacts such as peaking at the edges.

The frequency response of the Taylor window is

$$ab_k \begin{cases} = \frac{(\mathbf{m}'_{k+1})^2}{J_1(\mathbf{p} \cdot \mathbf{m}'_{k+1})} \cdot \frac{\prod_{n=1}^{\bar{n}-1} \left(1 - \left(\frac{\mathbf{m}'_{k+1}}{\mathbf{s} \cdot \mathbf{z}_{k+1}}\right)^2\right)}{\prod_{\substack{n=0 \\ n \neq k}}^{\bar{n}-1} \left(1 - \left(\frac{\mathbf{m}'_{k+1}}{\mathbf{m}'_{n+1}}\right)^2\right)}, & 0 \leq k < \bar{n} \\ = 0, & k \geq \bar{n} \end{cases} \quad (4.54)$$

and the weighting function itself is given by

$$ws(uxs, usy) = C \cdot usx \cdot \sum_{k=0}^{\bar{n}-1} ab_k \cdot J_1(\mathbf{p} \cdot us) \quad (4.55)$$

where usx , usy , and us are related to array coordinates according to Equation (4.9).

5. THREE AND MORE DIMENSIONS

First we note that the algorithms given in Chapter 3 will work in up to seven dimensions, the limit being on the number of subscripts allowable in FORTRAN. Extension of these FFTs to even higher numbers of dimensions is a trivial task, but is not deemed necessary at this time because requirements for eight or more dimensions is nonexistent for the time being, a user is likely to use his own code and quite possibly in another language, and eight dimensions with 16 points each is 4 billion complex data points, meaning that a data array using 32-bit floating point would occupy 32 gigabytes. Although problems of this size are not unheard of and will likely become important in the foreseeable future, limiting indices to 16 is not a good decision for most important problems. In summary, the examples in Chapter 3 will work as-is, at least for a first cut, for nearly all important problems – and they provide a basis for construction of user algorithms.

Applications using three or more dimensions include boundary value problems, synthetic aperture radar autofocus and mosaicing, true time-delay beamforming, and multiple preformed beams in sparse interferometry.

The Fourier-Bessel integral for spherically symmetric functions in K dimensions is, from Chapter 1,

$$F(\mathbf{r}) = \frac{2 \cdot \mathbf{p}^{\frac{K}{2}}}{\Gamma\left(\frac{N}{2}\right)} \cdot \int_0^R f(r) \cdot J_0(r \cdot \mathbf{r}) \cdot r^{K-1} \cdot dr \quad (4.56)$$

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