Contents

$$afg_k = \frac{1}{N^2} \cdot \sum_{i=0}^{N-1} \sum_{p=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} af_m \cdot ag_n^* \cdot \exp\left(-j \cdot \frac{2p}{N} \cdot \left(i \cdot k - p \cdot m + \left(p - i \right) \cdot n \right) \right) \tag{3.56}$$

Summation on i is possible because only complex exponentials are involved, and the result is a Dirichlet kernel, which is effectively a Kronecker delta scaled by N, or $N \cdot d_{k,n}$. This allows summation on n as well, with the only nonzero term on summation on n being the one for $n = k$. The result is

$$afg_k = \frac{1}{N} \cdot \sum_{p=0}^{N-1} \sum_{m=0}^{N-1} af_m \cdot ag_k^* \cdot \exp\left(-j \cdot \frac{2p}{N} \cdot p \cdot \left(-m + k \right) \right). \tag{3.57}$$

We can now sum on p and find another Dirichlet kernel, then sum on m to find

$$afg_k = af_k \cdot ag_k^* . \tag{3.58}$$

These and other results are summarized below as Table 2.

Table 1-2. DFT/FFT Transform Pairs

x_i	$a_k = \sum_{i=0}^{N-1} x_i \cdot \exp\left(-j \cdot \frac{2p}{N} \cdot i \cdot k \right)$	Remarks
a_k	$N \cdot x_{N-k}$	Double transform
$\sum_{p=0}^{N-1} f_p \cdot g_{i-p}^*$	$af_k \cdot ag_k^*$	Convolution, cross-correlation
$\sum_{p=0}^{N-1} f_p \cdot f_{i-p}^*$	$\left\| a_k \right\|^2$	Autocorrelation, Energy spectrum
$f_i \cdot g_i^*$	$\frac{1}{N} \cdot \sum_{p=0}^{N-1} af_p \cdot ag_{p-k}^*$	Multiplication, Convolution
$d_{i,p}$	$\exp\left(-j \cdot \frac{2p}{N} \cdot p \cdot k \right)$	Kronecker delta
$\exp\left(+j \frac{2p}{N} \cdot i \cdot s \right)$	$\frac{\sin\left(p \cdot (k-s) \right)}{\sin\left(\frac{p}{N} \cdot (k-s) \right)} \cdot \exp\left(-j \cdot \frac{p \cdot (N-1)}{N} \cdot (k-s) \right)$	Dirichlet kernel

6. GIBB'S PHENOMENON

Gibb's phenomenon is the behavior of an inverse Fourier transform, Fourier series, or DFT near a step discontinuity. For the Fourier transform, we see it when we look at the Fourier transform of the step function. For the Fourier transform, we come upon it looking at the unit step function in time,

Table 1-2. DFT/FFT Transform Pairs

x_i	$a_k = \sum_{i=0}^{N-1} x_i \cdot \exp\left(-j \cdot \frac{2p}{N} \cdot i \cdot k\right)$	Remarks
a_k	$N \cdot x_{N-k}$	Double transform
$\sum_{p=0}^{N-1} f_p \cdot g_{i-p}^{*}$	$af_k \cdot ag_k^{*}$	Convolution, cross-correlation
$\sum_{p=0}^{N-1} f_p \cdot f_{i-p}^{*}$	$\lvert a_k \rvert^2$	Autocorrelation, Energy spectrum
$f_i \cdot g_i^{*}$	$\frac{1}{N} \cdot \sum_{p=0}^{N-1} af_p \cdot ag_{p-k}^{*}$	Multiplication, Convolution
$d_{i,p}$	$\exp\left(-j \cdot \frac{2p}{N} \cdot p \cdot k\right)$	Kronecker delta
$\exp\left(+j \frac{2p}{N} \cdot i \cdot s\right)$	$\frac{\sin\left(p \cdot (k-s)\right)}{\sin\left(\frac{p}{N} \cdot (k-s)\right)} \cdot \exp\left(-j \cdot \frac{p \cdot (N-1)}{N} \cdot (k-s)\right)$	Dirichlet kernel

Note that our normalization of A differs from that of Taylor's paper for consistency with other paragraphs here. The number of "equiripple" sidelobes $\bar{n}$ must be large enough so that the broadening factor s is greater than one because the factor of s applies to the main lobe, and adding rolloff to the sidelobes cannot decrease main lobe width. Also, $\bar{n}$ must be large enough so that s decreases with increasing $\bar{n}$, because increasing the bandwidth over which the sidelobes are equiripple must decrease the main lobe width. This condition,

$$\bar{n} \geq \frac{1}{2} \cdot \left(4 \cdot \left(\frac{A}{p} \right)^2 + 1 \right) \tag{3.41}$$

is a hard limit on applicability – the frequency response of the Taylor window does resemble its desired shape when this condition is met, and the sidelobe structure is less clearly related to the design intent when this condition is violated. The minimax principle shows that, as Taylor stated in his original paper[50], the first $\bar{n}$ sidelobes cannot be down as far as designed unless s is greater than one, which can only be true when Equation (3.41) is satisfied.

We examine s as a function of $\bar{n}$, in the abstract. The broadening factor s is zero for $\bar{n}$ equal to zero, reaches a peak greater than one at the value given as the threshold in Equation (3.41), and decreases to an asymptote of one as $\bar{n}$ increases past that of the threshold value. This peak value is

$$s_{MAX} = \sqrt{1 + \frac{1}{4 \cdot \left(\frac{A}{p} \right)^2}} \tag{3.42}$$

which will be near one when A is large, that is to say when S is very large. Furthermore, decrease of s as $\bar{n}$ increases above the threshold value will be quite slow. Since the sidelobes of the frequency response are observed to be essentially as designed when Equation (3.41) is satisfied, practical designs are obtained by simply rounding up from the threshold value, or perhaps adding one or two to the threshold value before rounding. Increasing $\bar{n}$ much beyond that which is required to obtain the sidelobe heights will provide the frequency response shape according to the theory, but the behavior of the weighting function will begin to show artifacts such as peaking at the edges.

The frequency response of the Taylor window is

$$ab_k \begin{cases} = \dfrac{\left(m'_{k+1}\right)^2}{J_1\left(p \cdot m'_{k+1}\right)} \cdot \dfrac{\prod\limits_{n=1}^{\bar{n}-1}\left(1-\left(\dfrac{m'_{k+1}}{s \cdot z_{k+1}}\right)^2\right)}{\prod\limits_{\substack{n=0\\ n\neq k}}^{\bar{n}-1}\left(1-\left(\dfrac{m'_{k+1}}{m'_{n+1}}\right)^2\right)}, 0 \le k < \bar{n} \\ = 0, k \ge \bar{n} \end{cases} \tag{4.54}$$

and the weighting function itself is given by

$$ws\left(usx, usy\right) = C \cdot usx \cdot \sum_{k=0}^{\bar{n}-1} ab_k \cdot J_1\left(p \cdot us\right) \tag{4.55}$$

where usx, usy, and us are related to array coordinates according to Equation (4.9).

5. THREE AND MORE DIMENSIONS

First we note that the algorithms given in Chapter 3 will work in up to seven dimensions, the limit being on the number of subscripts allowable in FORTRAN. Extension of these FFTs to even higher numbers of dimensions is a trivial task, but is not deemed necessary at this time because requirements for eight or more dimensions is nonexistent for the time being, a user is likely to use his own code and quite possibly in another language, and eight dimensions with 16 points each is 4 billion complex data points, meaning that a data array using 32-bit floating point would occupy 32 gigabytes. Although problems of this size are not unheard of and will likely become important in the foreseeable future, limiting indices to 16 is not a good decision for most important problems. In summary, the examples in Chapter 3 will work as-is, at least for a first cut, for nearly all important problems – and they provide a basis for construction of user algorithms.

Applications using three or more dimensions include boundary value problems, synthetic aperture radar autofocus and mosaicing, true time-delay beamforming, and multiple preformed beams in sparse interferometry.

The Fourier-Bessel integral for spherically symmetric functions in K dimensions is, from Chapter 1,

$$F\left(r\right) = \frac{2 \cdot p^{\frac{K}{2}}}{\Gamma\left(\dfrac{N}{2}\right)} \cdot \int_0^R f\left(r\right) \cdot J_0\left(r \cdot r\right) \cdot r^{K-1} \cdot dr \tag{4.56}$$

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